ON ASYMMETRIC APPROXIMATIONS

by

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1. B. Segre [1] deduced the following theorem from his investigation of lattice points in certain non-convex domains:

Every irrational number ξ has infinitely many rational approximations u/v such that

$$(1) \qquad -\frac{\tau}{(1+4\tau)^{1/2}v^2} < \xi - \frac{u}{v} < \frac{1}{(1+4\tau)^{1/2}v^2}$$

where $\tau \geq 0$ is arbitrary.

C. D. Olds [2] gave a simple arithmetic proof for the case $\tau > 1$. N. Negoescu [3] used continued fractions to show that the inequality

$$(2) \qquad -\frac{\tau}{\alpha v^2} < \xi - \frac{u}{v} < \frac{1}{\alpha v^2}$$

has infinitely many solutions for $\tau \geq 0$ if $\alpha = \max((1+4\tau)^{\frac{1}{2}}, (\tau^2+4\tau)^{\frac{1}{2}})$, but as R. M. Robinson [5] pointed out, Segre's and Negoescu's theorems are equivalent, inasmuch as they are identical when $\tau \leq 1$, while for $\tau > 1$, Negoescu's theorem asserts the same property of ξ as does Segre's of $-\xi$, if τ is replaced by $1/\tau$ in (1). Recently, Negoescu [4] attempted to prove that of any three consecutive convergents of the continued fraction expansion of ξ, one at least satisfies (1) for arbitrary $\tau \geq 0$. It is shown here that this is true of one out of any five consecutive convergents; more precisely, at least one of the numbers p_{2n-1}/q_{2n-1}, p_{2n}/q_{2n}, p_{2n+1}/q_{2n-1} satisfies (2) with $\alpha = (1+4\tau)^{1/2}$, and one of the numbers p_{2n}/q_{2n}, p_{2n+1}/q_{2n+1}, p_{2n+2}/q_{2n+2} satisfies (2) with $\alpha = (\tau^2+4\tau)^{1/2}$. Here the p_k/q_k are convergents to ξ, n is an arbitrary positive integer, and $\tau > 0$ is arbitrary. Moreover, Negoescu's assertion is shown to be sometimes false.

For the special case $\tau = 1$ (or $c_1 = c_2$), the proof given in §2 simplifies considerably, and leads to a proof of the well-known theorem of Hurwitz whose perspicuity compares favorably with that given by A. Khintchine [6].

2. For $(p, q) = 1$, let $I(p/q)$ denote the closed interval