CONFORMAL MAPPINGS AND PEANO CURVES

by

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Salem and Zygmund [1945] have shown that if $\{n_k\}$ is a sequence of integers with the property $n_{k+1}/n_k \geq \lambda > \lambda_0$, where λ_0 is ome universal constant less than $1 + 10\sqrt{2}\,\pi$, then the function $\sum a_k z^{n_k}$ maps the unit circle into a Peano curve provided the series $\sum |a_k|$ converges so slowly that

$$\lambda^{1-p}|a_1| + \lambda^{2-p}|a_2| + \ldots + |a_p| < c \sum_{k>p} |a_k|$$

for $p = 1, 2, \ldots$; the constant c depends on λ; it is sufficient that it be less than

$$[\lambda(\lambda - 1) - 2^{3/2}\pi(5\lambda - 1)]/[\lambda(\lambda - 1) + 2^{3/2}\pi(5\lambda - 1)].$$

In the present note, we present a theorem which is contained in the result of Salem and Zygmund. Its publication is justified by the extreme simplicity of the proof.

Theorem. There exists a function $f(z)$ which is holomorphic in the open unit disc and continuous in the closed unit disc, and which has the property that the set of points $f(e^{i\theta})$ $(0 \leq \theta < 2\pi)$ fills a square.

The example by means of which the theorem will be proved is of the form

$$f(z) = \sum_{r=1} k_r[1 - (1 - z/t_r)^{\alpha_r}],$$

where $\{k_r\}$ is a complex null sequence, $\{t_r\}$ is a sequence of distinct points on the unit circle, and $\{\alpha_r\}$ is a sequence of positive constants approaching zero very rapidly; the function $(1 - r/t_r)^{\alpha_r}$ is understood to