THE NATURAL MAP OF THE HYPERBOLIC PLANE INTO THE EUCLIDEAN CIRCLE

by

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The non-conformal Euclidean circle model of the hyperbolic plane, with straight lines represented by chords, was introduced into the literature by Beltrami in 1868 (1) (the well-known "psuedo-sphere" appeared four years later) and it made a contribution toward the general acceptance of the new geometry, which was still widely regarded as somehow "unreal". Beltrami discovered this circle model in the course of an investigation of the differential geometry of a surface of constant negative curvature, treated intrinsically. The projective model, with conic as "absolute," was introduced by Klein and Cayley in 1871 (2)-1872 (3), and we now commonly think of the circle model as simply a special case of the projective. However, there exists a mapping of the hyperbolic plane into the Euclidean circle which is so simple and direct that it seems remarkable that it was not pointed out by one of the early non-Euclidean geometers. Had the modern concept of mathematical "model" been current at the time, it surely would have been noticed.

This mapping takes place in hyperbolic 3-space and makes use of the important theorem that the geometry on a horosphere ("limiting surface" = sphere of infinite radius) is Euclidean. It is defined as follows: In a hyperbolic 3-space of parameter p, let a plane m be tangent to a horosphere h at point O, and let each point of m map onto a point of h by projection along the corresponding axis of h.