ON FLOWS OF MEASURE-PRESERVING TRANSFORMATIONS

John Lamperti

1. INTRODUCTION

The purpose of this paper is to investigate some conditions under which a measure-preserving transformation can be embedded into a "flow" (real-parameter, measurable group) of transformations. The class of transformations to be studied consists of the invertible, measure-preserving, ergodic transformations of the unit interval onto itself which have "discrete spectrum;" that is, of transformations whose proper functions form a basis for L_2. (A convenient summary of terminology and facts concerning such transformations is contained in [2].) By a *d.s. transformation* will be meant a member of this class; its *spectrum* means the set of proper values (point spectrum). A d.s. transformation is determined to within conjugacy by its spectrum, which is a denumerable subgroup of the complex numbers of modulus 1 (and any countable subgroup of the unit circle is the spectrum of a d.s. transformation); therefore the embeddability criteria to be derived will be in terms of spectra.

Halmos has shown in [1] that (in particular) a d.s. transformation has a square root if and only if -1 does not belong to its spectrum. In Section 2, this result is generalized to give the criterion for the existence of nth roots. The result and the proof are analogous to the square root case. In Section 3, a necessary and sufficient condition is derived for the possibility of embedding a d.s. transformation into a flow. The condition is somewhat cumbersome; however, it has corollaries which show that the embedding is possible for many transformations, but that the existence of roots of all orders is not sufficient.

2. Nth ROOTS

LEMMA 1. *Let* G *be a countable subgroup of the unit circle, and* n *an integer* ($n \geq 2$). *In order that there exist another subgroup* G' *of the unit circle, isomorphic to* G *in such a way that the correspondence of* g' *to* g *implies that* $g'^n = g$, *it is necessary and sufficient that* G *contains no* n*th root of unity, except unity itself.*

The proof will be omitted, since it is not difficult and may be carried out in an analogous manner to the proof of Lemma 3 of [1].

THEOREM 1. *A d.s. transformation* T *has an* n*th root if and only if its spectrum contains no* n*th root of* 1 *other than* 1 *itself.*

Proof. Suppose that $T = R^n$. Then the spectrum of T consists of the nth powers of the spectrum of R. Both spectra are groups, and the correspondence of $g' \in Sp(R)$ to $g'^n = g \in Sp(T)$ is one-to-one, as a consequence of ergodicity. Hence, by the lemma, the spectrum of T contains no proper nth root of unity.

Now suppose that the spectrum of T contains no nth root of 1. Any d.s. transformation may be represented as (is conjugate to) a rotation of a compact Abelian group—the group may be taken to be the character group of the spectrum of the transformation, and then the element by which the group is rotated is the character

Received February 2, 1957.