ON MAPS OF THE THREE-SPHERE INTO THE PLANE

G. R. Livesay

1. INTRODUCTION

The following theorem deals with a special case of part of the Knaster conjecture [2].

THEOREM. *Let* $f: S^3 \to E^2$ *be continuous, and let* p, p_1, p_2 *be points of* S^3 *which are vertices of an equilateral triangle in* E^4. *Then there exists a rotation* $r \in SO(4)$ *such that* $f(rp) = f(rp_1) = f(rp_2)$.

This note consists of a proof of this theorem. Before giving the proof, let us fix the notation. E^n is Euclidean n-space, S^{n-1} the unit sphere of E^n; SO(n) is the group of proper rotations of E^n, considered here as operating on S^{n-1}; and P^n is real projective n-space. For $x \in S^3$, let G_x denote the subgroup of SO(4) consisting of rotations which leave x fixed. Let $f_1: S^3 \to E^1$ be the map obtained by following f by the projection of E^2 onto E^1 which is defined by the rule $(x_1, x_2) \to x_1$. Without loss of generality, suppose that p is a point of S^3 at which f_1 attains its maximum value, and that this maximum is positive. It is an elementary matter to show that there exists a rotation $a \in SO(4)$ satisfying

(1) $ap = p_1, \quad ap_1 = p_2, \quad a^3 = 1$ (where 1 denotes the identity element of SO(4)),

(2) a leaves some point, say $z \in S^3$, fixed.

Then G_p and G_z are conjugate subgroups of SO(4), and they carry homologous, non-bounding, integral 3-cycles of SO(4). The proof will now proceed as follows: we shall construct a map $\psi: SO(4) \to S^3$, under the assumption that the theorem is false. Then we shall see that $\psi | G_p$ and $\psi | G_z$ have different degrees. Since this is impossible, our proof by contradiction will then be complete.

2. CONSTRUCTION OF ψ

This construction is well known. Define the three maps

$$\phi: SO(4) \to E^6, \qquad T: SO(4) \to SO(4), \qquad T^1: E^6 \to E^6$$

by the conditions

$$\phi(r) = (f(rp), f(rp_1), f(rp_2)),$$

$$T(r) = r \cdot a,$$

$$T^1(x_1, \cdots, x_6) = (x_3, x_4, x_5, x_6, x_1, x_2).$$

Then $\phi \circ T = T^1 \circ \phi$, since $ap = p_1$, and so forth. Let

Received June 8, 1957.

The author wishes to thank the Office of Naval Research for their support of this research.