On the Definition of Clifford Algebras

by

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Clifford algebras are usually defined in one of two ways. Let K be a field of characteristic not two. One method is to give a basis of the algebra [1]. The basis consists of the elements e_A where A ranges through the subsets of the set $N = \{1, 2, \ldots, n\}$, including the null set $\emptyset$. We write e_i for $e_{\{i\}}$ and define

(1) $e_i^2 = a_i e_\emptyset \quad (i = 1, \ldots, n)$

where the a_i are elements of K; also

(2) $e_i e_j = -e_j e_i \quad (i \neq j)$.

Then if $A = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$, we require that $e_A = e_{i_1} \cdots e_{i_r}$ and $e_\emptyset = 1$. From (1) and (2) products of the e_A can be defined. That multiplication is associative needs to be verified by computation.

A second method of definition is more intrinsic [2]. Let V be an n-dimensional vector space over K. Let T(V) be the tensor algebra of V, i.e., the free associative algebra over K consisting of sums of products of vectors in V, where it is assumed that the product with a scalar is commutative. Let f be a symmetric bilinear scalar function on V. Let J be the ideal of T(V) generated by all $vw + wv - 2f(v, w)$, where v and w range through V. The difference algebra $T(V)/J$ is defined to be a Clifford algebra.