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Determinant Functors on Exact Categories and Their Extensions to Categories of Bounded Complexes

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Introduction

In this paper I revisit a theme unsatisfactorily treated in [KM]. The methods used here are more natural and more general. The theorem we prove was suggested to me by Grothendieck in a letter dated May 19, 1973 (see Appendix B), and it states that the category of determinants on the derived category of an exact category is equivalent via restriction to the category of determinants on the exact category itself.

Here is how the problem comes about [KM]. Consider the following category. The objects are bounded complexes of locally free finite quasi-coherent sheaves of $\mathcal{O}_X$-modules on a fixed scheme (site) X. The morphism $\mathrm{Mor}(A, B)$ of two such complexes is the group of global sections of the sheaf of germs of homotopy classes of homomorphisms from A to B. If we assign to every complex the invertible sheaf

$$f(A) = \left(\bigotimes_{i\in\mathbb{Z}} \bigwedge^{\max} A^{2i}\right) \otimes \left(\bigotimes_{i\in\mathbb{Z}} \bigwedge^{\max} A^{2i+1}\right)^{-1},$$

then the problem is to assign to every quasi-isomorphism $\alpha \in \mathrm{Mor}(A, B)$ an isomorphism $f(\alpha)\colon f(A) \to f(B)$ in such a way that f becomes a functor and such that $f = \bigwedge^{\max}$ in case of a complex consisting of a single locally free sheaf supported in degree 0. The existence of such an f follows immediately from the theorem. The theorem is quite general and depends (a) on certain properties of projective modules over a *commutative* ring and short exact sequences of such, and (b) on certain properties of tensor products of modules of rank 1.

The appropriate notions are that of an exact category (see [Q, Sec. 2]) and that of a commutative Picard category. The reader not familiar with the notion of an exact category is advised to have in mind the category of finitely generated *projective* modules over a commutative ring, where exact sequences are what they are. An admissible monomorphism is an injection whose cokernel is projective, and similarly an admissible epimorphism is a surjection with projective kernel. Of course, in this particular case all surjections are admissible.

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