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L^2 Harmonic Forms for a Class of Complete Kähler Metrics

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1. Introduction

The Hodge theorem for compact manifolds states that every real cohomology class of a compact manifold M is represented by a unique harmonic form. That is, the space of solutions to the differential equation $(d+d^*)\phi = 0$ on L^2 forms over M, a space that depends on the metric on M, is canonically isomorphic to the purely topological real cohomology space of M. This isomorphism is enormously useful because it provides a way to transform theorems from geometry into theorems in topology and vice versa. No such result holds in general for complete noncompact manifolds, but in many specific cases there are Hodge-type theorems. One of the oldest is the description, due to Atiyah, Patodi, and Singer [1], of the space of L^2 harmonic forms on a manifold with complete cylindrical ends. By calculating the solutions to the equation for harmonic forms on the cylindrical ends, they showed that the space of L^2 harmonic forms is isomorphic to the image of the relative cohomology of the manifold in the absolute cohomology. Another Hodge-type result was found by Zucker [14] for a natural class of metrics called Poincaré metrics. These metrics, first constructed by Cornalba and Griffiths [4], are complete Kähler metrics with hyperbolic cusp-type singularities at isolated points on a Riemann surface. Zucker showed that the space of L^2 forms on a Riemann surface that are harmonic with respect one of these metrics is isomorphic to the standard cohomology of the surface. This result was extended by Cattani, Kaplan, and Schmid [3] to analogous metrics on bundles over projective varieties with singularities along a divisor. These metrics can be thought of as complete Kähler metrics on the noncompact manifold given by removing the divisor.

There is a larger natural class of complete, noncompact Kähler metrics on the complement of a divisor in a projective variety. They are of interest both because of their relation to the Poincaré metrics and because other examples of them have arisen in papers by Tian and Yau [12; 13] as starting points for the construction of metrics solving the Kähler–Einstein problem, and the final metrics in these papers are quasi-isometric to the starting metrics. In this paper, we study the space of L^2 harmonic forms on manifolds with such metrics and its relation to cohomology of the original projective variety, especially with a view to how the spaces relate to Hodge diamond structures on subspaces of the cohomology of the original variety.

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