Michigan Math. J. 50 (2002)

Some Applications of Bruhat–Tits Theory to Harmonic Analysis on the Lie Algebra of a Reductive p-adic Group

Jeffrey D. Adler & Stephen DeBacker

1. Introduction

In recent years, the questions of interest in the study of harmonic analysis on reductive p-adic groups have required very precise versions of what were previously qualitative results (see e.g. [19; 20; 25; 26; 27]). This paper began as an attempt to prove precise versions of some results of Fiona Murnaghan that relate the character of a supercuspidal representation to the Fourier transform of an elliptic orbital integral [15; 16; 17; 18]. In order to properly formulate these results, it was necessary to develop a "uniform" way to express both the support of invariant distributions and the local constancy of functions. We present here the product of this effort.

Let F denote a field with discrete valuation. We assume that F is complete with perfect residue field $\mathfrak{f}$. Let G be the group of F-rational points of a reductive, connected, linear algebraic group defined over F, and let $\mathfrak{g}$ denote its Lie algebra. Let $\mathcal{B}$ denote the Bruhat–Tits building of G.

Recall that, for $x \in \mathcal{B}$ and $r \in \mathbf{R}$, Allen Moy and Gopal Prasad defined a lattice $\mathfrak{g}_{x,r}$ of $\mathfrak{g}$. In Section 3 we explore the relationship between the lattices $\mathfrak{g}_{x,r}$ and $\mathcal{N}$, the set of nilpotent elements in $\mathfrak{g}$. For every real number r we construct the open, closed, G-invariant subset

$$\mathfrak{g}_r := \bigcup \mathfrak{g}_{x,r},$$

where the union is taken over the points in $\mathcal{B}$. The sets $\mathfrak{g}_r$ can be used to describe the support of invariant distributions on $\mathfrak{g}$. We show that

$$\mathfrak{g}_r = \bigcap (\mathfrak{g}_{x,r} + \mathcal{N}),$$

where the intersection is taken over the points in $\mathcal{B}$. This equality provides some intuition for the ubiquity of the nilpotent set in harmonic analysis. We prove that the sets $\mathfrak{g}_r$ behave well with respect to parabolic descent. That is, if P is a parabolic subgroup of G with Levi decomposition $P = MN$ and Lie algebras $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$, then

$$\mathfrak{m} \cap \mathfrak{g}_r = \mathfrak{m}_r.$$

Received May 17, 2001. Revision received January 31, 2002.

The first-named author was partially supported by a University of Akron Faculty Research Grant (#1470).