Michigan Math. J. 50 (2002)

On Certain Loci of Smooth Degree $d \geq 4$ Plane Curves with d-Flexes

G. Casnati & A. Del Centina

0. Introduction and Notation

Vermeulen [Ve] studied the subvarieties $\mathcal{V}_\alpha \subseteq \mathfrak{M}_3$ (where $\mathfrak{M}_g$ is the moduli space of smooth, genus-g curves over the complex field $\mathbb{C}$) corresponding to plane, smooth quartics C having α hyperflexes—that is, pairs $h := (P, r) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2$ (with $\mathbb{P}^n$ the projective n-space over $\mathbb{C}$) such that $C \cdot r = 4P$.

Vermeulen proved that if $\alpha = 1, 2$ then $\mathcal{V}_\alpha$ is an irreducible subvariety of dimension $6 - \alpha$, and that $\mathcal{V}_3$ is the union of three irreducible components each of whose dimension is 3. He also studied $\mathcal{V}_\alpha$ for $\alpha \geq 4$. Since it follows from the results listed in [Ve] that each component of $\mathcal{V}_\alpha$ is unirational, we obtain that all such components are actually rational when $\alpha \geq 4$ (via Castelnuovo's and Lüroth's theorems, since their dimension is at most 2).

The aim of this paper is to generalize these kind of results by considering smooth, plane curves of degree d having d-flexes, that is, pairs $h := (P, r) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2$ such that $C \cdot r = dP$.

Let $d \geq 4$ and $g := \binom{d-1}{2}$, and denote by $\mathcal{V}_{d,\alpha} \subseteq \mathfrak{M}_g$ the locus of points representing isomorphism classes of smooth, plane curves having α d-flexes. In Section 2 we prove the following theorem.

Theorem A. *The loci $\mathcal{V}_{d,\alpha}$ ($\alpha = 1, 2$) are irreducible, rational locally closed subvarieties of dimension $\binom{d+2-\alpha}{2} - 8 + 3\alpha$.*

The locus $\mathcal{V}_{4,1}$ has been considered also by Faber [Fa], who proved that the Chow ring $A(\mathfrak{M}_3)$ can be generated by it together with the hyperelliptic locus $\mathcal{H}_3$. Since $\mathcal{H}_3$ is known to be rational (see [BK] and [Ka]; see also [Do] and [PV]), it follows that $A(\mathfrak{M}_3)$ can be generated by rational subvarieties (see [CD] for a similar result about $A(\mathfrak{M}_4)$).

When $\alpha \geq 3$, the locus $\mathcal{V}_{d,\alpha}$ is no longer irreducible. Let $\{h_i\}_{i=1,\dots,\alpha}$ be the set of d-flexes of C. For any triple in this set, one can define a projective invariant $\Lambda_{i,j,k} := \lambda(h_i, h_j, h_k)$ that always satisfies $\Lambda^d_{i,j,k} = 1$ (see Section 1). In order to detect the irreducible components of $\mathcal{V}_{d,\alpha}$, one introduces the $\binom{\alpha}{3}$-tuple

Received January 31, 2001. Revision received September 25, 2001.

Both the authors have been partially supported by MURST in the framework of the national project "Geometria Algebrica, Algebra Commutativa ed Aspetti Computazionali".