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Comparison of the Pluricomplex and the Classical Green Functions on Convex Domains of Finite Type

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1. Introduction

Let D be a bounded domain with Lipschitz boundary in $\mathbf{R}^n$, and let y be a fixed point in D. Then there is a solution $h_y(x)$ to the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0 & \text{in } D, \\ u(x) = -\eta(x-y) & \text{on } \partial D, \end{cases}$$

where

$$\eta(x) = \begin{cases} \log|x| & \text{if } N = 2, \\ -|x|^{2-N} & \text{if } N \geq 3. \end{cases}$$

The function $G_D(x,y) = \eta(x-y) + h_y(x)$ is called the *classical* (*negative*) *Green function* for the Laplacian, with pole at y. It is harmonic in $D \backslash \{y\}$ and tends to zero on the boundary; furthermore, it is symmetric.

Now let D be a bounded domain in $\mathbf{C}^n$. By $\mathrm{PSH}(D)$ we denote the class of plurisubharmonic (psh) functions on D. The *pluricomplex Green function* for D with pole at w is defined by

$$g_D(z,w) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}(D),\ \varphi \leq 0,\ \varphi(z) \leq \log|z-w| + O(1)\}.$$

This definition was first given by Klimek [5]. It coincides with the classical Green function in the complex plane. The function $g_D(\cdot, w)$ is a negative plurisubharmonic function in D and has a logarithmic pole at w. It is decreasing with respect to holomorphic maps, which implies that it is biholomorphically invariant. If D is hyperconvex, then $g_D(z,w) \to 0$ as $z \to \partial D$ and g_D is continuous on $\bar{D} \times D$ (cf. [3]). The pluricomplex Green function is symmetric for convex domains [7], although it is not symmetric in general [1]. The pluricomplex Green function plays a similar role in the pluripotential theory as the classical Green function in the classical potential theory, so it is interesting to compare the two. In the case when D is strongly pseudoconvex, Carlehed [2] proved that the following holds for all $z, w \in D$:

$$\frac{g_D(z,w)}{G_D(z,w)} \leq C(D)|z-w|^{2n-4}.$$

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