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Moduli Spaces of Vector Bundles on Higher-Dimensional Varieties

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1. Introduction

Let X be an n-dimensional, smooth, irreducible, algebraic variety over $\mathbb{C}$ and let L be an ample divisor on X. Let $M_{X,L}(r; c_1, \dots, c_{\min(r,n)})$ denote the moduli space of rank-r, L-stable (in the sense of Mumford and Takemoto) vector bundles E on X with Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$. Moduli spaces for stable vector bundles on smooth, irreducible, algebraic projective varieties were constructed in the 1970s. Many interesting results have been proved regarding these moduli spaces when the underlying variety is a surface, but very little is known if the variety has dimension greater than or equal to three. Until now there have been no general results about these moduli spaces concerning the number of connected components, dimension, smoothness, rationality, topological invariants, and so forth.

A major result in the theory of vector bundles on an algebraic surface S was the proof that, for large c_2, $M_{S,L}(r; c_1, c_2)$ is irreducible, generically smooth, and of the expected dimension $2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(O_S)$. For moduli spaces of vector bundles on a higher-dimensional variety, the situation differs drastically. The smoothness and irreducibility turn out to be false when $\dim X \geq 3$. For instance, in [BM, Thm. 0.1], Ballico and Miró-Roig prove that, under certain technical restrictions on c_1, the number of irreducible components of the moduli space $M_{X,L}(2; c_1, c_2)$ of L-stable, rank-2 vector bundles on a smooth projective 3-fold X, with fixed c_1 and $c_2 L$ going to infinity, grows to infinity. See [MO] for examples of singular moduli spaces of vector bundles on $\mathbb{P}^{2n+1}$ with $c_2 \gg 0$.

Let $X = \mathbb{P}(\mathcal{E}) \to C$ be a $\mathbb{P}^d$-bundle over a smooth projective curve C of genus $g \geq 0$. The goal of this paper is to compute the dimension, prove the irreducibility and smoothness, and describe the structure of the moduli space $M_{X,L}(2; c_1, c_2)$ for a suitable polarization L closely related to c_2. More precisely, we will cover the study of all moduli spaces $M_{X,L}(2; c_1, c_2)$ such that the general point $[E] \in M_{X,L}(2; c_1, c_2)$ is given as a nontrivial extension of line bundles (Theorems 3.4, 3.5, 3.8, and Remark 3.9). In particular, for rational normal scrolls (i.e., $\mathbb{P}^d$-bundles over $\mathbb{P}^1$) and for a certain choice of c_1, c_2 and L, we have that the moduli space $M_{X,L}(2; c_1, c_2)$ is rational (Corollary 3.6). Therefore, the geometry of the underlying variety and of the moduli spaces are intimately

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