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# Families of Affine Planes: The Existence of a Cylinder

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## Introduction

Dolgachev and Weisfeiler [9, (3.8.5)] formulated the following.

Conjecture. *Let $f: X \to S$ be a flat affine morphism of smooth schemes with every fiber isomorphic* (*over the residue field*) *to an affine space. Then $f$ is locally trivial in the Zariski topology.*

In the characteristic-0 case, this conjecture is known to be true (under much weaker assumptions) for morphisms of relative dimension 1 ([24; 23, Thm. 2]; see also [30, Thm. 2] and [5; 6; 10]). Another proof based on the Rosenlicht–Chevalley–Grothendieck theory of special algebraic groups [2; 37] was indicated by Danilov; see [9]. The known partial positive results in higher relative dimensions (see e.g. [30; 38] and [4, (3.9)–(3.10)]) deal only with families over a 1-dimensional base with 2-dimensional fibers, under an extra assumption that the generic fiber is the affine plane as well. In this paper we show that the latter assumption holds over any base. To simplify consideration, we restrict it to smooth, quasi-projective varieties defined over $\mathbb{C}$ (actually, Theorem 0.1 remains true over any algebraically closed field of characteristic 0).

We say that a family $f: X \to S$ of quasi-projective varieties *contains a cylinder* if, for some Zariski open subset $S_0$ of $S$, there is a commutative diagram

$$\begin{tikzcd} f^{-1}(S_0) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & S_0 \times \mathbb{C}^k \arrow[dl, "\mathrm{pr}_1"] \\ & S_0 & \end{tikzcd}$$

where $\varphi$ is an isomorphism. (In general, by a *cylinder over $U$* we mean a Cartesian product $U \times \mathbb{C}^k$ where $k > 0$.)

Our main result is the following theorem.

Theorem 0.1. *A smooth family $f: X \to S$ with general fibers isomorphic to $\mathbb{C}^2$ contains a cylinder $S_0 \times \mathbb{C}^2$.*

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