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Brauer Equivalence in a Homogeneous Space with Connected Stabilizer

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0. Introduction

In this note we investigate the Brauer equivalence in a homogeneous space $X = G/H$, where G is a simply connected semisimple algebraic group over a local field or a number field and H is a connected subgroup of G.

In more detail, let k be a field of characteristic 0, and let $\bar{k}$ be a fixed algebraic closure of k. For a smooth algebraic variety Y over k, set $\bar{Y} = Y_{\bar{k}} = Y \times_k \bar{k}$. Let $\operatorname{Br} Y$ denote the cohomological Brauer group of Y, $\operatorname{Br} Y = H^2_{\text{ét}}(Y, \mathbb{G}_m)$. Set $\operatorname{Br}_1 Y = \ker[\operatorname{Br} Y \to \operatorname{Br} \bar{Y}]$. There is a canonical pairing

$$Y(k) \times \operatorname{Br}_1 Y \to \operatorname{Br} k, \quad (y, b) \mapsto b(y) \tag{0.1}$$

called the *Manin pairing.* We define the Brauer equivalence on $Y(k)$ as follows: $y_1 \sim y_2$ if $(y_1, b) = (y_2, b)$ for all $b \in \operatorname{Br}_1 Y$. We denote the set of classes of Brauer equivalence in $Y(k)$ by $Y(k)/\operatorname{Br}$. Note that we define the Brauer equivalence in terms of $\operatorname{Br}_1 Y$, not in terms of $\operatorname{Br}_1 Y^c$ or $\operatorname{Br} Y^c$, where Y^c is a smooth compactification of Y.

The notion of B-equivalence for a subgroup B of the Brauer group $\operatorname{Br} Y$ was introduced by Manin [M1; M2]. Colliot–Thélène and Sansuc [CS1] investigated the Brauer equivalence in algebraic tori (they defined the Brauer equivalence in terms of the Brauer group of a smooth compactification). The Brauer equivalence in reductive groups was studied in [T].

Let G be a simply connected semisimple algebraic group over k. Let H be a connected subgroup of G. We denote by H^{tor} the biggest toric quotient group of H. We are interested in the Brauer equivalence in the set $X(k)$ where $X = G/H$.

We compute $X(k)/\operatorname{Br}$ when k is a local field. Namely, we prove that there is a bijection

$$X(k)/\operatorname{Br} \xrightarrow{\sim} \operatorname{im}\bigl[\ker[H^1(k, H) \to H^1(k, G)] \to H^1(k, H^{\text{tor}})\bigr]$$

(Theorem 2.1). Moreover, when k is a non-archimedean local field, we prove that there is a bijection $X(k)/\operatorname{Br} \xrightarrow{\sim} H^1(k, H^{\text{tor}})$ (Theorem 2.2).

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