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C^1 Immersed Hypersurfaces Separate $\mathbb{R}^n$

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1. Introduction

Clearly an immersed 2-sphere separates $\mathbb{R}^3$. As evidenced by the papers of Vaccaro, Feighn, and M. D. Hirsch, this statement is true—but it is less than clear. We summarize the known results for proper immersions $f : M^m \to N^n$ where the codimension $k = n - m$ is ≥ 1 and M, N are boundaryless. Recall that a map is *proper* if pre-images of compact sets in the target are compact in the domain.

(a) Vaccaro [7]. If f is merely PL (piecewise linear) then everything fails; the counterexample is the house with two rooms, which is a nonseparating PL immersion of the 2-sphere in $\mathbb{R}^3$. The illustration in Figure 1 is drawn as piecewise smooth. See also Rourke and Sanderson [6] or Bing [1].

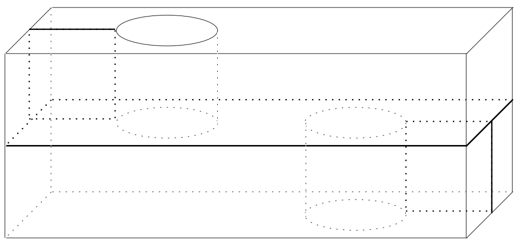

Figure 1 The house with two rooms

(b) Vaccaro [7]. If f is C^1 and if fM is a subcomplex of a C^1 triangulation of N then $H_m(fM; \mathbb{Z}_2) \neq 0$, which by Alexander duality implies that fM separates when $N = \mathbb{R}^{m+1}$.

(c) Feighn [2]. If f is C^2, $k = n - m = 1$, and $H_1(N; \mathbb{Z}_2) = 0$, then fM separates N.

(d) Hirsch [4]. If f is C^2 then fM *k-separates* N in the sense that the kth homology and the kth homotopy groups of the pair $(N, N \setminus fM)$ are nontrivial. The coefficient group for the homology can be either $\mathbb{Z}$ or $\mathbb{Z}_2$.

Note that k-separation is also referred to as "k-piercing."

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