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Mappings of Finite Distortion: Condition N

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1. Introduction

Suppose that f is a continuous mapping from a domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) into $\mathbb{R}^n$. We consider the following Lusin condition N: If $E \subset \Omega$ with $\mathcal{L}^n(E) = 0$, then $\mathcal{L}^n(f(E)) = 0$. Physically, this condition requires that there be no creation of matter under the deformation f of the n-dimensional body Ω. This is a natural requirement, since the N property with differentiability a.e. is sufficient for validity of various change-of-variable formulas, including the area formula, and the condition N holds for a homeomorphism f if and only if f maps measurable sets to measurable sets.

If the coordinate functions of f belong to the Sobolev class $W^{1,1}_{\text{loc}}(\Omega)$ and $|Df| \in L^p(\Omega)$ for some $p > n$, then f satisfies the Lusin condition N (Marcus and Mizel, [14]). Recently we verified in [10] that this also holds when $|Df|$ belongs to the Lorentz space $\mathrm{L}^{n,1}(\Omega)$ and that this analytic assumption is essentially sharp even if the determinant of Df is nonnegative a.e. For a homeomorphism, less regularity is needed: it suffices to assume that $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$; this is due to Reshetnyak [19]. On the other hand, there exists a homeomorphism that does not satisfy the condition N and so $|Df|$ belongs to $L^p(\Omega)$ for each $p < n$; see the examples by Ponomarev [17; 18]. Some further results on the Lusin condition are listed in the survey paper [13].

We will need the concept of topological degree. We say that a continuous mapping f is *sense-preserving* if the topological degree with respect to any subdomain $G \subset\subset \Omega$ is strictly positive: $\deg(f, G, y) > 0$ for all $y \in f(G) \setminus f(\partial G)$. In this paper we show that, for a sense-preserving mapping, the sharp regularity assumption in the rearrangement-invariant scale to rule out the failure of the condition N is that

$$\lim_{\varepsilon \to 0+} \varepsilon \int_{\Omega} |Df|^{n-\varepsilon} = 0. \tag{1.1}$$

Theorem A. *Suppose that $f : \Omega \to \mathbb{R}^n$ is sense-preserving and that* (1.1) *holds. Then f satisfies condition N. On the other hand, there is a homeomorphism f from the closed unit cube Q_0 onto Q_0 such that*

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