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Sharp Weighted Endpoint Estimates for Commutators of Singular Integrals

Carlos Pérez & Gladis Pradolini

1. Introduction and Statements of the Main Result

The main purpose of this paper is to improve the main result in [P2] by means of a direct proof that avoids the classical good-λ technique considered there. The good-λ method, introduced by Burkholder and Gundy in [BG], is a powerful tool but has the disadvantage that it is essentially adapted to measures satisfying the A_∞ condition, such as the Lebesgue measure. The approach we consider here is more related to the classical argument of Calderón and Zygmund for proving that singular integral operators satisfy the weak-type $(1,1)$-property, an approach whose advantage is that it allows us to consider more general measure. The method, however, must be different because commutators of singular integral operators with BMO functions are not of weak-type $(1,1)$, as shown in [P2].

Let b be a locally integrable function on $\mathbf{R}^n$, usually called the *symbol*, and let T be a Calderón–Zygmund singular integral operator (see [C] or [J]). Consider the commutator operator $[b,T]$ defined for, say, smooth functions f by

$$[b,T]f = bT(f) - T(bf). \tag{1}$$

A now classical result of Coifman, Rochberg, and Weiss [CRW] states that $[b,T]$ is a bounded operator on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, when b is a BMO function. In fact, BMO is also a necessary condition for the commutator $[b,R]$ to be bounded on $L^p(\mathbf{R}^n)$, where $R = (R^1,\dots,R^n)$ is the vector-valued Riesz transform. We will always assume that $b \in \mathrm{BMO}(\mathbf{R}^n)$ unless otherwise noted.

None of the different proofs of this result follows the usual scheme of the classical Calderón–Zygmund theory of singular integral operators T. Indeed, the key result in this theory is that any of these operators satisfies the weak-type $(1,1)$-property, which is derived from the assumption that T is bounded on $L^2(\mathbf{R}^n)$ combined with a mild regularity of the kernel. Once the weak-type $(1,1)$-inequality is obtained, interpolation and duality yield the boundedness of the operator on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$. However, simple examples show that commutators (with BMO symbols) fail to be of weak-type $(1,1)$, as found in [P2]. To remedy

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