Michigan Math. J. 72 (2022), 449–463
https://doi.org/10.1307/mmj/20207202

Simple Supercuspidals and the Langlands Correspondence

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1. Formal Degrees

In the fall of 2007, Mark Reeder and I were thinking about the formal degrees of discrete series representations of p-adic groups G. The formal degree is a generalization of the dimension of an irreducible representation, when G is compact. It depends on the choice of a Haar measure dg on G: if the irreducible representation π is induced from a finite dimensional representation W of an open, compact subgroup K, then the formal degree of π is given by $\deg(\pi) = \dim(W)/\int_K dg$.

Mark had determined the formal degree in several interesting cases [19], including the series of depth zero supercuspidal representations that he had constructed with Stephen DeBacker [2]. We came across a paper by Kaoru Hiraga, Atsushi Ichino, and Tamotsu Ikeda, which gave a beautiful conjectural formula for the formal degree in terms of the L-function and ε-factor of the adjoint representation of the Langlands parameter [13, §1]. This worked perfectly in the depth zero case [13, §3.5], and we wanted to test it on more ramified parameters, where the adjoint L-function is trivial.

Naturally, we started with the group $\mathrm{SL}_2(\mathbb{Q}_p)$. In the course of our work, Mark found a new construction of irreducible representations, which we called simple supercuspidals. His discovery, when viewed through the prism of the Langlands correspondence, has led to some interesting mathematics. I want to survey this here.

Finally, I am delighted to submit this paper to a volume in honor of Gopal Prasad, whose ideas have guided my work in so many ways. In particular, the Moy–Prasad filtration of the Iwahori subgroup led us to the correct definition of simple supercuspidals in the general case.

2. The Conjecture

The formula that Hiraga, Ichino, and Ikeda propose for the formal degree of a discrete series representation π depends on the (still conjectural) Langlands parametrization of irreducible representations. We will state this when $\mathbf{G}$ is a split, simply connected group defined over the ring of integers A of a p-adic field k, and $G = \mathbf{G}(k)$ is the locally compact group of points with open compact subgroup $K = \mathbf{G}(A)$. Let F be the residue field of A, which is finite of order q. As in [6]

Received October 8, 2020. Revision received April 30, 2021.