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Maximal Newton Points and the Quantum Bruhat Graph

Elizabeth Milićević

Abstract. We discuss a surprising relationship between the partially ordered set of Newton points associated with an affine Schubert cell and the quantum cohomology of the complex flag variety. The main theorem provides a combinatorial formula for the unique maximum element in this poset in terms of paths in the quantum Bruhat graph, whose vertices are indexed by elements in the finite Weyl group. Key to establishing this connection is the fact that paths in the quantum Bruhat graph encode saturated chains in the strong Bruhat order on the affine Weyl group. This correspondence is also fundamental in the work of Lam and Shimozono establishing Peterson's isomorphism between the quantum cohomology of the finite flag variety and the homology of the affine Grassmannian. One important geometric application of the present work is an inequality which provides a necessary condition for nonemptiness of certain affine Deligne–Lusztig varieties in the affine flag variety.

1. Introduction

This paper investigates connections between the geometry and combinatorics in two different, but surprisingly related contexts: certain subvarieties of the affine flag variety in characteristic $p > 0$ and the quantum cohomology of the complex flag variety. The main results establish explicit relationships among fundamental questions in both theories, using paths in the quantum Bruhat graph as the primary dictionary. We begin with a brief historical survey of these two geometric contexts in order to frame the informal statement of the main theorem.

1.1. Newton Polygons

In the 1950s, Dieudonné introduced the notion of isocrystals over perfect fields of characteristic $p > 0$ (see [Man63]), which Grothendieck extended to families of F-crystals in [Gro74]. Isogeny classes of F-crystals are indexed by combinatorial objects called *Newton polygons,* a partially ordered set of lattice polygons in the plane. Kottwitz used the machinery of algebraic groups to explicitly study the set of Newton points associated with any connected reductive group G over a discretely valued field in [Kot85; Kot97]. In particular, he observed that there

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