Michigan Math. J. 69 (2020), 153–178

On Gromov–Witten Theory of Projective Bundles

Honglu Fan & Yuan-Pin Lee

0. Introduction

0.1. Statement of the Main Result

Let Y be a smooth projective variety with a T-action, and let V be a T-equivariant, *not necessarily split,* vector bundle of rank r over Y. The projective bundle $\mathbb{P}(V)$ naturally carries an induced T-action. The equivariant cohomology of $\mathbb{P}(V)$ has the following presentation:

$$H_T^*(\mathbb{P}(V)) = \frac{H_T^*(Y)[h]}{(c_T(V)(h))},$$

where $h = c_1(\mathcal{O}(1))$ and $c_T(V)(x) = \sum x^{r-i} c_i(V)$. Given two such equivariant vector bundles V_1, V_2 over Y with the same equivariant Chern classes $c_T(V_1) = c_T(V_2)$, the equivariant cohomology/Chow are canonically isomorphic by the above presentation

$$\mathfrak{F} : H_T^*(\mathbb{P}(V_1)) \cong H_T^*(\mathbb{P}(V_2)).$$

The isomorphism $\mathfrak{F}$ induces an isomorphism between the groups of numerical curve classes $N_1(\mathbb{P}(V_1))$ and $N_1(\mathbb{P}(V_2))$ by the intersection pairing, and we slightly abuse the notation and denote the induced isomorphism also by $\mathfrak{F}$. This isomorphism on curve classes is uniquely characterized by the property $(\mathfrak{F}D, \mathfrak{F}\beta) = (D, \beta)$ for any $D \in N^1(\mathbb{P}(V_1))$, $\beta \in N_1(\mathbb{P}(V_1))$.

Theorem A (=Theorem 4.2). *If Y is a projective smooth variety with a torus action such that there are finitely many fixed points and one-dimensional orbits, then the $\mathfrak{F}$ induces an isomorphism of T-equivariant genus 0 Gromov–Witten invariants between $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$:*

$$\langle \psi^{a_1}\sigma_1, \dots, \psi^{a_n}\sigma_n\rangle_{0,n,\beta}^{\mathbb{P}(V_1)} = \langle \psi^{a_1}\mathfrak{F}\sigma_1, \dots, \psi^{a_n}\mathfrak{F}\sigma_n\rangle_{0,n,\mathfrak{F}\beta}^{\mathbb{P}(V_2)}.$$

Such Y is often called a proper algebraic GKM manifold, and examples include toric varieties, Grassmannians, flag varieties, and certain Hilbert schemes of points.

Received January 17, 2018. Revision received August 17, 2018.

Both authors are partially supported by the NSF. The first author is also supported by SwissMAP and ERC-2012-AdG-320368-MCSK in the group of Rahul Pandharipande at ETH Zürich.