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On Certain Complex Projective Manifolds with Hodge Numbers $h^{10} = 4$ and $h^{20} = 5$

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0. Introduction

Throughout this paper, Z denotes a closed connected complex submanifold of $\mathbb{P}^N_{\mathbb{C}}$ that satisfies the following:

ASSUMPTION 0.1. (i) The Hodge numbers of Z satisfy $h^{10}(Z) = 4$ and $h^{20}(Z) = 5$.
(ii) The image of an Albanese map $\alpha(Z) \subset \mathrm{Alb}(Z)$ has dimension ≥ 2.
(iii) $\mathrm{Alb}(Z)$ is a simple Abelian variety.

The first main result is that there are strong restrictions on the possible endomorphism algebras of the Albanese variety of Z, $E(\mathrm{Alb}(Z)) := \mathrm{End}(\mathrm{Alb}(Z)) \otimes \mathbb{Q}$.

THEOREM 0.2. *$E(\mathrm{Alb}(Z))$ does not have type I in Albert's classification of endomorphism algebras of simple Abelian varieties. If $E(\mathrm{Alb}(Z))$ has type II, then $\dim_{\mathbb{Q}}(E(\mathrm{Alb}(Z))) \neq 8$. If $E(\mathrm{Alb}(Z))$ has type IV, then $\dim_{\mathbb{Q}}(E(\mathrm{Alb}(Z))) \neq 4$.*

Often a much finer invariant of an Abelian variety than its endomorphism algebra is its Hodge group. However, in the following circumstance, if the endomorphism algebra is known, then so is the Hodge group.

THEOREM 0.3. *If $E = E(\mathrm{Alb}(Z))$ is an imaginary quadratic number field, then the Hodge group of $\mathrm{Alb}(Z)$ is the special unitary group of a four-dimensional Hermitian E-vector space. Furthermore, the Hodge group is quasi-split.*

These results place strong restrictions on the period map of any irreducible family of projective varieties that includes Z.

The paper is organized as follows. After notations are introduced in Section 1 and a basic lemma is proved in Section 2, the three statements in Theorem 0.2 are proved in Sections 3, 4, and 5. The last of these sections also contains the proof of Theorem 0.3. Section 6 recalls a construction of minimal surfaces of general type satisfying Assumption 0.1 and establishes that the endomorphism algebra of the Albanese variety may have any of the types II, III, or IV. In Section 7 the notion of an Albanese exotic variety is recalled, and a new criterion for Albanese exoticity is given. The usefulness of this criterion is illustrated in the last two sections where

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