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Almost Abelian Artin Representations of $\mathbb{Q}$

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Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. All number fields considered here are understood to be subfields of $\overline{\mathbb{Q}}$. We write $\mathbb{Q}^{\mathrm{ab}}$ for the maximal abelian extension of $\mathbb{Q}$ and $\mathbb{Q}^{\mathrm{aa}}$ for the maximal almost abelian extension, the latter being defined as the compositum of all finite Galois extensions K of $\mathbb{Q}$ such that the commutator subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ is central of exponent dividing 2. Note that $\mathbb{Q}^{\mathrm{ab}} \subset \mathbb{Q}^{\mathrm{aa}}$. Anderson [1] has proved the following beautiful complement to the Kronecker–Weber theorem:

$$\mathbb{Q}^{\mathrm{aa}} = \mathbb{Q}^{\mathrm{ab}}\bigl(\bigl\{\sqrt[4]{\ell} : \ell \text{ prime}\bigr\} \cup \bigl\{\sqrt{t_{p,q}} : p, q \text{ prime}, p < q\bigr\}\bigr), \tag{1}$$

where if p is odd then $t_{p,q} = s_{p,q}/s_{q,p}$ with

$$s_{p,q} = \prod_{j=1}^{(p-1)/2} \left(\frac{\sin(\pi j/p)}{\prod_{k=0}^{(q-1)/2} \sin(\pi(j+pk)/(pq))} \right),$$

while if $p = 2$ then

$$t_{p,q}^{-1} = 2^{q/2} \left(\prod_{k=0}^{(q-1)/2} \sin\left(\pi \frac{1+4k}{4q}\right)\right) \times \left(\prod_{j=1}^{(q-1)/2} \frac{\sin(\pi j/q)\sin(\pi(2j-1)/(2q))}{\sin(\pi j/(2q))\sin(\pi(2j-1)/(4q))} \right).$$

Although we have departed from Anderson's notation slightly, our $t_{p,q}$ nonetheless coincides with Anderson's $\sin \mathbf{a}_{pq}$.

In this note, we use Anderson's work to establish a connection between *almost abelian Artin representations* of $\mathbb{Q}$—in other words, Artin representations of $\mathbb{Q}$ that factor through $\mathrm{Gal}(\mathbb{Q}^{\mathrm{aa}}/\mathbb{Q})$—and *Hecke–Shintani representations.* The latter term refers to two-dimensional irreducible monomial Artin representations of $\mathbb{Q}$ that can be induced from more than one quadratic field. The intended allusion is to Shintani's work [12] on Stark's conjecture, which rests on the fact that certain irreducible two-dimensional Artin representations of $\mathbb{Q}$ induced from real quadratic fields can also be induced from imaginary quadratic fields, making it possible to deduce Stark's conjecture in such cases from the Kronecker limit formula. However, Shintani himself credits Hecke ([12], p. 158): "A coincidence of an L-series of a real quadratic field with an L-series of an imaginary quadratic field was first observed by Hecke." In any case, we shall see that Hecke–Shintani

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