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# Rational Curves on Hypersurfaces

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Abstract. Let $(X, D)$ be a pair where $X$ is a projective variety. We study in detail how the behavior of rational curves on $X$ and the positivity of $-(K_X + D)$ and $D$ influence the behavior of rational curves on $D$. In particular, we give criteria for uniruledness and rational connectedness of components of $D$.

## 1. Introduction

For a projective variety $X$, the connection between the positivity of $-K_X$ and the behavior of rational curves on $X$ is well understood. Uniruledness and rational connectedness are possibly two birational properties of smooth varieties that have been the most intensively studied. A result of Miyaoka and Mori [MM86] shows that a smooth projective variety $X$ is uniruled if and only if there exists a $K_X$-negative curve through every general point of $X$. Later Boucksom, Demailly, Păun, and Peternell [BDPP13] proved that if the canonical divisor of a projective manifold $X$ is not pseudoeffective, then $X$ is uniruled. The rational connectedness of smooth Fano varieties was established by Campana [Cam92] and Kollár, Miyaoka, and Mori [KMM92], and it was later generalized to the log Fano cases by Zhang [Zha06] and Hacon and McKernan [HM07].

A natural question is how the behavior or rational curves on a variety $X$ influences the behavior of rational curves on a hypersurface $D$. An easy case is where $X = \mathbb{P}^n$; then a general hypersurface of degree $\leq n$ is rationally connected. More generally, if $(X, D)$ is a plt pair and $-(K_X + D)$ is ample, then by the adjunction formula we have $(K_X + D)|_D = K_D + \mathrm{Diff}_D(0)$, which is antiample and klt. So by [Zha06, Theorem 1] $D$ is rationally connected and, in particular, uniruled. However, if we assume that $-(K_X + D)$ is big and semiample instead of ample, then the following example shows that $D$ is not necessarily uniruled.

Example 1.1. Let $\pi : X = \mathbb{P}(\mathcal{E}) \to C$ be a ruled surface, where $C$ is an elliptic curve, and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ is such that $\mathcal{L}$ is a line bundle on $C$ and $\deg(\mathcal{L}) < 0$. Let $e = -\deg(\bigwedge^2 \mathcal{E})$. Then $e > 0$ and $K_X \equiv_{\mathrm{num}} -2C_0 - eF$, where $C_0$ is the unique section of $\pi$ with $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$ (see [Har77, Chapter V, Example 2.11.3]), and $F$ is a fiber. So we have

$$-(K_X + C_0) \equiv_{\mathrm{num}} C_0 + eF = \varepsilon C_0 + (1-\varepsilon)\left(C_0 + \frac{e}{1-\varepsilon}F\right),$$

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