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SO(n) Covariant Local Tensor Valuations on Polytopes

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Abstract. The Minkowski tensors are valuations on the space of convex bodies in $\mathbb{R}^n$ with values in a space of symmetric tensors, having additional covariance and continuity properties. They are extensions of the intrinsic volumes, and as these, they are the subject of classification theorems and admit localizations in the form of measure-valued valuations. For these local tensor valuations, restricted to convex polytopes, a classification theorem has been proved recently under the assumption of isometry covariance, but without any continuity assumption. This characterization result is extended here, replacing the covariance under orthogonal transformations by invariance under proper rotations only. This yields additional local tensor valuations on polytopes in dimensions two and three, but not in higher dimensions. In this paper, they are completely classified.

1. Introduction

A valuation on the space $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ is a mapping φ from $\mathcal{K}^n$ into some Abelian group such that

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$. The best known examples are the intrinsic volumes or Minkowski functionals. They arise as the suitably normalized coefficients of the polynomial in ρ that expresses, for a given convex body K, the volume of the outer parallel body of K at distance $\rho \geq 0$. The celebrated characterization theorem of Hadwiger states that every rigid motion invariant continuous real-valued valuation on $\mathcal{K}^n$ is a linear combination of the intrinsic volumes. This theorem was the first culmination of a rich theory of valuations on convex bodies (for the older history, see the surveys [15] and [17]), which in the last two decades has again been widened and deepened considerably. For an introduction and for references, we refer to [24], in particular, Chapter 6 and Section 10.16. A survey on recent developments is given by Alesker [3].

A natural extension of the intrinsic volumes is obtained if the volume is replaced by a higher moment. If the integral

$$\int_K x \otimes \cdots \otimes x \, \mathrm{d}x,$$

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