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Chern–Ricci Invariance Along G-Geodesics

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Abstract. Over a compact oriented manifold, the space of Riemannian metrics and normalized positive volume forms admits a natural pseudo-Riemannian metric G, which is useful for the study of Perelman's $\mathcal{W}$ functional. We show that if the initial speed of a G-geodesic is G-orthogonal to the tangent space to the orbit of the initial point under the action of the diffeomorphism group, then this property is preserved along all points of the G-geodesic. We show also that this property implies preservation of the Chern–Ricci form along such G-geodesics under the extra assumption of complex antiinvariant initial metric variation and vanishing of the Nijenhuis tensor along the G-geodesic.

1. Statement of the Invariance Result

We consider the space $\mathcal{M}$ of smooth Riemannian metrics over a compact oriented manifold X of dimension m. We denote by $\mathcal{V}_1$ the space of positive smooth volume forms with integral one. Notice that the tangent space of $\mathcal{M} \times \mathcal{V}_1$ is

$$T_{\mathcal{M}\times\mathcal{V}_1} = C^\infty(X, S^2T_X^*) \oplus C^\infty(X, \Lambda^m T_X^*)_0,$$

where $C^\infty(X, \Lambda^m T_X^*)_0 := \{V \in C^\infty(X, \Lambda^m T_X^*) \mid \int_X V = 0\}$. We denote by $\mathrm{End}_g(T_X)$ the bundle of g-symmetric endomorphisms of T_X and by $C_\Omega^\infty(X, \mathbb{R})_0$ the space of smooth functions with zero integral with respect to Ω. We will use the fact that, for any $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$, the tangent space $T_{\mathcal{M}\times\mathcal{V}_1,(g,\Omega)}$ identifies with $C^\infty(X, \mathrm{End}_g(T_X)) \oplus C_\Omega^\infty(X, \mathbb{R})_0$ via the isomorphism

$$(v, V) \longmapsto (v_g^*, V_\Omega^*) := (g^{-1}v, V/\Omega).$$

In [Pal4], we consider the pseudo-Riemannian metric G over $\mathcal{M} \times \mathcal{V}_1$, defined over any point $(g, \Omega) \in \mathcal{M} \times \mathcal{V}_1$ by the formula

$$G_{g,\Omega}(u, U; v, V) = \int_X [\langle u, v\rangle_g - 2U_\Omega^* V_\Omega^*]\Omega$$

for all $(u, U), (v, V) \in T_{\mathcal{M}\times\mathcal{V}_1}$. The gradient flow of Perelman's $\mathcal{W}$-functional [Per] with respect to the structure G is a modification of the Ricci flow with relevant properties (see [Pal4; Pal5]). The G-geodesics exists only for short time intervals $(-\varepsilon, \varepsilon)$. This is because the G-geodesics are uniquely determined by the evolution of the volume forms and the latter degenerate in finite time (see Section 2). In [Pal4], we show that the space G-orthogonal to the tangent of the

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