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Simplicial Embeddings Between Multicurve Graphs

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Abstract. We study some graphs associated to a surface, called k-multicurve graphs, which interpolate between the curve complex and the pants graph. Our main result is that, under certain conditions, simplicial embeddings between multicurve graphs are induced by π_1-injective embeddings of the corresponding surfaces. We also prove the rigidity of the multicurve graphs.

1. Introduction

Since the introduction of graphs associated to surfaces, there have been different questions raised about them. One line of questions concerns maps between such graphs. In particular, as the mapping class group usually acts on these complexes by automorphisms, it is interesting to understand whether these are all automorphisms or there are some not induced by a self-homeomorphism of the underlying surface. This is the *rigidity* problem: a graph associated with a surface is *rigid* if its automorphism group is the mapping class group of its surface. The first proof of the rigidity of the curve graph is due to Ivanov [Iva97] in the case of surfaces of genus at least two; subsequently, Korkmaz [Kor99] proved it for low-genus surfaces and Luo [Luo00] proved it in the general case. Margalit [Mar04] proved the rigidity of the pants graph by reducing it to the rigidity of the curve complex. Multiple other graphs have been proven to be rigid (like the Hatcher–Thurston graph [Irm06] and the graph of nonseparating curves [IK07]), and it is conjectured that graphs associated with surfaces having no obvious obstructions (such as being disconnected) should be rigid.

A more subtle problem is studying *simplicial embeddings* (i.e. injective maps preserving the graph structure) between such objects. There are embeddings that correspond to maps between the underlying surfaces: for instance, in the case of the curve graph, a π_1-injective embedding of a surface into another (which corresponds to seeing the first surface as an essential subsurface of the second) induces a simplicial embedding between the curve graphs. A similar construction holds for the pants graph: suppose S_1 is a subsurface of S_2 and choose a k-multicurve ν in the complement of the subsurface, where k is the difference of the complexities of the surfaces. Then we get a simplicial embedding of the pants graph of S_1 into the pants graph of S_2 by sending a pants decomposition μ of S_1 to $\mu \cup \nu$. Aramayona [Ara10] showed that, except in some low-complexity cases, any simplicial embedding between pants graphs arises this way. On the other hand, for

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