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Equivariantly Uniformly Rational Varieties

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Abstract. We introduce equivariant versions of uniform rationality: given an algebraic group G, a G-variety is called G-uniformly rational (resp. G-linearly uniformly rational) if every point has a G-invariant open neighborhood equivariantly isomorphic to a G-invariant open subset of the affine space endowed with a G-action (resp. linear G-action). We establish a criterion for $\mathbb{G}_m$-uniform rationality of smooth affine varieties equipped with hyperbolic $\mathbb{G}_m$-actions with a unique fixed point, formulated in terms of their Altmann–Hausen presentation. We prove the $\mathbb{G}_m$-uniform rationality of Koras–Russell threefolds of the first kind, and we also give an example of a non-$\mathbb{G}_m$-uniformly rational but smooth rational $\mathbb{G}_m$-threefold associated with pairs of plane rational curves birationally nonequivalent to a union of lines.

Introduction

A *uniformly rational* variety is a variety for which every point has a Zariski open neighborhood isomorphic to an open subset of an affine space. A uniformly rational variety is in particular a smooth rational variety, but the converse is an open question [10, p. 885].

In this article, we introduce stronger equivariant versions of this notion, in which we require in addition that the open subsets are stable under certain algebraic group actions. The main motivation is that for such varieties, uniform rationality, equivariant or not, can essentially be reduced to rationality questions at the quotient level. We construct examples of smooth rational but not equivariantly uniformly rational varieties; the question of their uniform rationality is still open. We also establish equivariant uniform rationality of large families of affine threefolds.

We focus mainly on actions of algebraic tori $\mathbb{T}$. The *complexity* of a $\mathbb{T}$-action on a variety is the codimension of a general orbit; in the case of a faithful action, the complexity is thus simply $\dim(X) - \dim(\mathbb{T})$. The complexity zero corresponds to toric varieties, which are well known to be uniformly rational when smooth. In fact, they are even $\mathbb{T}$-linearly uniformly rational in the sense of Definition 4. The same conclusion holds for smooth rational $\mathbb{T}$-varieties of complexity one by a result of [18, Chapter 4]. In addition, by [3, Theorem 5] any smooth complete rational $\mathbb{T}$-variety of complexity one admits a covering by finitely many open charts isomorphic to the affine space.

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