Michigan Math. J. 66 (2017), 203–219

$(p-1)$th Roots of Unity mod p^n, Generalized Heilbronn Sums, Lind–Lehmer Constants, and Fermat Quotients

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Abstract. For $n \geq 3$, we obtain an improved estimate for the generalized Heilbronn sum $\sum_{x=1}^{p-1} e_{p^n}(yx^{p^{n-1}})$ and use it to show that any interval $\mathcal{I}$ of points in $\mathbb{Z}_{p^n}$ of length $|\mathcal{I}| \gg p^{1.825}$ for $n = 2$, $|\mathcal{I}| \gg p^{2.959}$ for $n = 3$, and $|\mathcal{I}| \geq p^{n-3.269(34/151)^n+o(1)}$ for $n \geq 4$ contains a $(p-1)$th root of unity. As a consequence, we derive an improved estimate for the Lind–Lehmer constant for the Abelian group $\mathbb{Z}_p^n$ and improved estimates for Fermat quotients.

1. Introduction

Let p be a prime, $n \in \mathbb{N}$, $\mathbb{Z}_{p^n}^*$ be the group of units mod p^n, and $G_n \subset \mathbb{Z}_{p^n}^*$ be the subgroup of $(p-1)$th roots of unity,

$$G_n := \{x \in \mathbb{Z}_{p^n}^* : x^{p-1} = 1\} = \{x^{p^{n-1}} \pmod{p^n} : 1 \leq x \leq p-1\}.$$

For $y \in \mathbb{Z}$, let $S_n(y)$ denote the generalized Heilbronn sum

$$S_n(y) := \sum_{x \in G_n} e_{p^n}(yx) = \sum_{x=1}^{p-1} e_{p^n}(yx^{p^{n-1}}),$$

where $e_{p^n}(\cdot) = e^{\frac{2\pi i \cdot}{p^n}}$, and let

$$H_n = \max_{p^n \nmid y} |S_n(y)|.$$

Our interest here is in estimating H_n and studying the distribution of points in G_n. In particular, we wish to determine how large M must be so that any interval

$$\mathcal{I} := \{a+1, \ldots, a+M\} \subset \mathbb{Z}_{p^n} \tag{1.1}$$

of length M is guaranteed to contain an element of G_n. Equivalently, we wish to determine an upper bound on the maximal gap between consecutive $(p-1)$th roots of unity. It is well known that an estimate for H_n leads to a corresponding estimate on the size of the gap. We make this explicit in Corollary 3.1, where we prove that any interval of length $|\mathcal{I}| \geq 3p^{n-1}H_n$ contains an element of G_n.

The current best estimate for H_2 is due to Shkredov [17, Thm. 15],

$$H_2 \ll p^{\frac{5}{6}} \log^{\frac{1}{6}} p, \tag{1.2}$$

Received November 12, 2015. Revision received November 29, 2016.