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The Chow Ring of a Fulton–MacPherson Compactification

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Abstract. We give a short proof of a presentation of the Chow ring of the Fulton–MacPherson compactification of n points on an algebraic variety. The result can be found already in Fulton and MacPherson's original paper. However, there is an error in one of the lemmas used in their proof. In the process we also determine the Chow rings of weighted Fulton–MacPherson compactifications.

1. Introduction

For a topological space X, let $F(X, n)$ denote the configuration space of n distinct ordered points on X. In a seminal paper, Fulton and MacPherson [FM94] studied the question of how the space $F(X, n)$ can be *compactified* in the special case where X is a smooth projective algebraic variety. This question may at first seem absurd: what could be nicer than the obvious inclusion $F(X, n) \hookrightarrow X^n$? However, in algebraic geometry we often wish to compactify an open variety in such a way that it becomes the complement of a divisor with normal crossings. To this end, they proposed a different compactification denoted $X[n]$, now called the *Fulton–MacPherson compactification.*

Just like X^n, the space $X[n]$ admits a modular interpretation, where the boundary parameterizes certain "degenerate" configurations of points on X. However, instead of allowing points to collide, the space $X[n]$ is set up so that the variety X itself is allowed to degenerate in a controlled manner. The effect is that when points try to come together, X acquires a new irreducible component, a projective space of the appropriate dimension, on which the points end up and remain distinct. See [FM94, pp. 194–195] for a more precise description. They show that the boundary $X[n] \setminus F(X, n)$ will indeed be a strict normal crossing divisor, that the combinatorial structure of the boundary strata admits a pleasant combinatorial description in terms of rooted trees, and that $X[n]$ can be constructed from X^n by an explicit sequence of blow-ups in smooth centers.

Their construction is related to (and was inspired by) the Deligne–Mumford compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ of the moduli space of smooth curves of genus g with n distinct ordered points. In fact, the fiber of $\mathcal{M}_{g,n} \to \mathcal{M}_g$ over a moduli point $[X]$ is the configuration space $F(X, n)$, and the fiber of $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ over the same point is the Fulton–MacPherson compactification $X[n]$. Moreover,

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