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Relative Asymptotics for General Orthogonal Polynomials

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Abstract. In this paper, we study right limits of the Bergman shift matrix. Our results have applications to ratio asymptotics, weak asymptotic measures, relative asymptotics, and zero counting measures of orthogonal and orthonormal polynomials. Of particular interest are the applications to random orthogonal polynomials on the unit circle and real line.

1. Introduction

Let μ be a positive probability measure with compact and infinite support in the complex plane. Given such a measure, it is well known how to form the sequence of orthonormal polynomials for this measure, which we denote by $\{p_n(z;\mu)\}_{n\geq 0}$, satisfying

$$\int p_n(z;\mu)\overline{p_m(z;\mu)}\,d\mu(z)=\delta_{m,n}.$$

The polynomial $p_n(z;\mu)$ has a positive leading coefficient, which we denote by $\kappa_n(\mu)$. The monic polynomial $\kappa_n(\mu)^{-1}p_n(z;\mu)$ will be denoted by $P_n(z;\mu)$ and is of interest in its own right.

It is both interesting and informative to try to understand the relationship between the measure μ and the corresponding sequence of orthonormal polynomials. There are many ways that we can study the orthogonal polynomial asymptotics. One of the most interesting kinds of asymptotic behavior is known as *ratio asymptotics* and concerns the following limits:

$$\lim_{n\to\infty}\frac{p_{n-1}(z;\mu)}{p_n(z;\mu)}, \tag{1}$$

$$\lim_{n\to\infty}\frac{P_{n-1}(z;\mu)}{P_n(z;\mu)}, \tag{2}$$

provided that these limits exist. When studying ratio asymptotics, we are concerned with the existence of the limit, the domain of the limit, and a functional form of the limit.

A second kind of asymptotic behavior is known as *weak asymptotic measures* and concerns the weak limits of the measures $\{|p_n(z;\mu)|^2\,d\mu(z)\}_{n\in\mathbb{N}}$ as $n\to\infty$. Since μ has compact support, weak limits always exist, and we can gain insight into properties of the measure μ by understanding these weak limits. In some

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