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Belyi's Theorem for Complete Intersections of General Type

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ABSTRACT. We give a Belyi-type characterization of smooth complete intersections of general type over $\mathbb{C}$ that can be defined over $\overline{\mathbb{Q}}$. Our proof uses the higher-dimensional analogue of the Shafarevich boundedness conjecture for families of canonically polarized varieties, finiteness results for maps to varieties of general type, and rigidity theorems for Lefschetz pencils of complete intersections.

1. Introduction

The aim of this paper is to prove a Belyi-type characterization of smooth complete intersections of general type over $\mathbb{C}$ that can be defined over $\overline{\mathbb{Q}}$. In other words, we generalize Belyi's theorem for curves to certain higher-dimensional varieties. To motivate this, let us briefly discuss applications of Belyi's theorem for curves.

Belyi's theorem for curves was famously used by Grothendieck to show that the action of the absolute Galois group of $\mathbb{Q}$ on the set of Galois dessins is faithful [35, Thm. 4.7.7]. This result started a flurry of activity on dessins d'enfants [32]. Subsequently, the action of the Galois group of $\mathbb{Q}$ on the set of connected components of the coarse moduli space of surfaces of general type was proven to be faithful by Bauer, Catanese, and Grunwald [2]; see also the work of Easton and Vakil [9] and González-Diez and Torres-Teigell [14].

To state our main theorem, let $\mathbb{Q} \to \overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let X be a smooth projective connected curve over $\mathbb{C}$. If X can be defined over $\overline{\mathbb{Q}}$, then Belyi [4] proved that there exists a morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ ramified over at most three points. Conversely, by classical results of Weil and Grothendieck [15; 19; 42], if there exists a nonconstant morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ ramified over precisely three points, then X can be defined over $\overline{\mathbb{Q}}$. In other words, X can be defined over $\overline{\mathbb{Q}}$ if and only if X admits a rational function with at most three critical points.

The main result of this paper states that a smooth complete intersection X of general type over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if X admits a Lefschetz function with only three critical points (see Definition 4.1). Here by a Lefschetz function on X is meant a rational function $X \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ on X that factors via a Lefschetz pencil $X \dashrightarrow \mathbb{P}^1_{\mathbb{C}}$ and a rational function $\mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ on the projective line.

THEOREM 1.1. *Let X be a smooth complete intersection of general type over $\mathbb{C}$. Then the following are equivalent*:

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