Twisting of Composite Torus Knots

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Abstract. We prove that the family of connected sums of torus knots $T(2, p) \# T(2, q) \# T(2, r)$ is nontwisted for any odd positive integers $p, q, r \geq 3$, partially answering in the positive a conjecture of Tera-gaito [19].

1. Introduction

Let $K$ be a knot in the 3-sphere $S^3$, and $D^2$ a disk intersecting $K$ in its interior. Let $n$ be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes $K$ into a new knot $K_n$ in $S^3$. Let $\omega = \text{lk}(\partial D^2, L)$. We say that $K_n$ is obtained from $K$ by $(n, \omega)$-twisting (or simply twisting). Then we write $K \xrightarrow{(n,\omega)} K_n$ or $K \xrightarrow{(n,\omega)} K(n,\omega)$. We say that $K_n$ is an $(n, \omega)$-twisted knot (or simply a twisted knot) if $K$ is the unknot (see Figure 1).

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2, 3)$ is obtained from the unknot $T(2, 1)$ by a $(+1, 2)$-twisting (in this case, $n = +1$ and $\omega = +2$). A less obvious example is given in Figure 3, where it is shown that the composite knot $T(2, 3) \# T(2, 5)$ can be obtained from the unknot by a $(+1, 4)$-twisting (in this case, $n = +1$ and $\omega = +4$); see [10]. Here, $T(2, q)$ denotes the $(2, q)$-torus knot (see [11]).

Active research on twisting of knots started around 1990. One pioneer was the author’s Ph.D. thesis advisor Y. Mathieu, who asked the following questions in [13].

QUESTION 1.1. Is every knot in $S^3$ twisted? If not, what is the minimal number of twisting disks?

QUESTION 1.2. Is every twisted knot in $S^3$ prime?

To answer Question 1.1, Miyazaki and Yasuhara [15] were the first to give an infinite family of knots that are nontwisted. In particular, they showed that the granny knot, that is, the product of two right-handed trefoil knots, is the smallest nontwisted knot. In his Ph.D. thesis [3], the author showed that $T(5, 8)$ is the smallest nontwisted torus knot. This was followed by a joint work with Yasuhara [4], in which we gave an infinite family of nontwisted torus knots (i.e., $T(p, p + 7)$ for any $p \geq 7$) using some techniques derived from old gauge theory. On the other hand, Ohyama [16] showed that any knot in $S^3$ can be untied by (at most) two disks.

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