Michigan Math. J. **66 (2017), 21–35**

A Note on Higher-Order Gauss Maps

Sandra Di Rocco, Kelly Jabbusch, & Anders Lundman

Abstract. We study Gauss maps of order k, associated to a projective variety X embedded in projective space via a line bundle L. We show that if X is a smooth, complete complex variety and L is a k-jet spanned line bundle on X, with $k \geq 1$, then the Gauss map of order k has finite fibers, unless $X = \mathbb{P}^n$ is embedded by the Veronese embedding of order k. In the case where X is a toric variety, we give a combinatorial description of the Gauss maps of order k, its image, and the general fibers.

1. Introduction

Let $X \subset \mathbb{P}^N$ be an n-dimensional irreducible, nondegenerate projective variety defined over an algebraically closed field $\Bbbk$ of characteristic 0. The (classical) Gauss map is the rational morphism $\gamma : X \dashrightarrow \mathrm{Gr}(n, N)$ that assigns to a smooth point x the projective tangent space of X at x, $\gamma(x) = \mathbb{T}_{X,x} \cong \mathbb{P}^n$. It is known that the general fiber of γ is a linear subspace of $\mathbb{P}^N$ and that the morphism is finite and birational if X is smooth unless X is all of $\mathbb{P}^N$, [Zak93; KP91; GH79].

In [Zak93], Zak defines a generalization of the above definition as follows. For $n \leq m \leq N-1$, let $\mathrm{Gr}(m, N)$ be the Grassmanian variety of m-dimensional linear subspaces in $\mathbb{P}^N$, and define $\mathcal{P}_m = \overline{\{(x, \alpha) \in X_{sm} \times \mathrm{Gr}(m, N) \mid \mathbb{T}_{X,x} \subseteq L_\alpha\}}$, where L_α is the linear subspace corresponding to $\alpha \in \mathrm{Gr}(m, N)$, and the bar denotes the Zariski closure in $X \times \mathrm{Gr}(m, N)$. The *mth Gauss map* is the projection $\gamma_m : \mathcal{P}_m \to \mathrm{Gr}(m, N)$. When $m = n$, we recover the classical Gauss map, $\gamma_n = \gamma$. These generalized Gauss maps still enjoy the property that a general fiber is a linear subspace [Zak93, 2.3(c)]. Moreover, a general fiber is always finite if X is smooth and $n \leq m \leq N - n + 1$, [Zak93, 2.3(b)].

In this paper we consider a different generalization of the Gauss map, where, instead of higher-dimensional linear spaces tangent at a point, we use linear spaces tangent to higher order, namely the osculating spaces. The osculating space of order k of X at a smooth point $x \in X_{sm}$, $\mathbf{Osc}_x^k$, is a linear subspace of $\mathbb{P}^N$ of dimension d_k, where $n \leq d_k \leq \binom{n+k}{n}$; see Definition 2.4. We can then define a rational map $\gamma^k : X \dashrightarrow \mathrm{Gr}(d_k - 1, N)$ that assigns to a point x the kth osculating space of X at x, $\gamma^k(x) = \mathbf{Osc}_x^k$, where d_k is the general kth osculating dimension; see Definition 3.1. Notice that when $k = 1$, we recover the classical Gauss map, $\gamma^1 = \gamma_n = \gamma$. We call γ^k the *Gauss map of order k*. This definition was originally

Received June 23, 2015. Revision received October 9, 2016.
The first and third authors were partially supported by the VR grants [NT:2010-5563, NT:2014-4763].
The second author was partially supported by the Göran Gustafsson foundation.