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A Remark on FI-Module Homology

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Abstract. We show that the FI-homology of an FI-module can be computed via a Koszul complex. As an application, we prove that the Castelnuovo–Mumford regularity of a finitely generated torsion FI-module is equal to its degree.

1. Introduction

Let R be a commutative ring. Let FI be the category of finite sets and injective maps. An FI-module is a (covariant) functor from FI to the category of R-modules. In [3, (11)], Church, Ellenberg, Farb, and Nagpal defined, for any FI-module V, a complex $\widetilde{S}_{-*}V$ of FI-modules. The same complex was also considered independently by Putman [7, §4], who called it the central stability chain complex. The purpose of our present paper is to give a proof that the homology of the complex $\widetilde{S}_{-*}V$ is the FI-homology $H_*^{\mathrm{FI}}(V)$ of V, whose definition we now recall.

Let $\mathbb{N}$ be the set of nonnegative integers. For each $n \in \mathbb{N}$, let $[n]$ be the set $\{1, \dots, n\}$ with n elements; in particular, $[0] = \emptyset$. It is convenient to introduce the nonunital R-algebra

$$A := \bigoplus_{0 \le m \le n} A_{m,n},$$

where $A_{m,n}$ is the free R-module on the set $\mathrm{Hom}_{\mathrm{FI}}([m],[n])$. For any $\alpha \in \mathrm{Hom}_{\mathrm{FI}}([m],[n])$ and $\beta \in \mathrm{Hom}_{\mathrm{FI}}([r],[s])$, their product $\alpha\beta$ in A is defined by

$$\alpha\beta := \begin{cases} \alpha \circ \beta & \text{if } s = m, \\ 0 & \text{else.} \end{cases}$$

Define a two-sided ideal J of A by

$$J := \bigoplus_{0 \le m < n} A_{m,n}.$$

For each $n \in \mathbb{N}$, let $e_n \in A_{n,n}$ be the identity endomorphism of $[n]$. A left A-module M is said to be *graded* if $M = \bigoplus_{n \in \mathbb{N}} e_n M$. If V is an FI-module, then $\bigoplus_{n \in \mathbb{N}} V([n])$ has the natural structure of a graded left A-module. The functor $V \mapsto \bigoplus_{n \in \mathbb{N}} V([n])$ is an equivalence from the category of FI-modules to the category of graded left A-modules. Thus, we shall not distinguish between the notion

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