Michigan Math. J. 65 (2016), 813–832

Rigidity of the Strongly Separating Curve Graph

Brian H. Bowditch

Abstract. We define the strongly separating curve graph to be the full subgraph of the curve graph of a compact orientable surface, where the vertex set consists of all separating curves that do not bound a three-holed sphere. We show that, for all but finitely many surfaces, any automorphism of the strongly separating curve graph is induced by an element of the mapping class group.

1. Introduction

The main aim of this paper is to prove a rigidity result (Theorem 1.1) for certain curve graphs associated to compact orientable surfaces. It is a variation on some well-known results in this direction. Our main motivation for this particular statement is its application to the quasi-isometric rigidity of the Weil–Petersson metric.

Let Σ be a compact orientable surface. We write $g(\Sigma)$ for its genus, and $p(\Sigma)$ for the number of boundary components. The *complexity* $\xi(\Sigma)$ of Σ is defined by $\xi(\Sigma) = 3g(\Sigma) + p(\Sigma) - 3$. (It equals the number of disjoint simple closed curves needed to cut Σ into a collection of 3-holed spheres.)

Let $\mathcal{G}(\Sigma)$ be the curve graph associated to Σ, that is, the 1-skeleton of the curve complex as defined in [H]. It has vertex set $C(\Sigma)$, the set of nontrivial nonperipheral simple closed curves in Σ, defined up to homotopy. Two elements of $C(\Sigma)$ are deemed adjacent if they can be homotoped to be disjoint. Note that the mapping class group $\mathrm{Map}(\Sigma)$ acts cofinitely on $\mathcal{G}(\Sigma)$. The rigidity theorems of [Iv; Ko; L] tell us (in particular) that if $\xi(\Sigma) \geq 2$, then any automorphism of $\mathcal{G}(\Sigma)$ is induced by an element of $\mathrm{Map}(\Sigma)$. (Note that since the curve complex is a flag complex, this is equivalent to the same statement for the curve complex.)

There are a number of variations of this. Given a subset $A \subseteq C(\Sigma)$ we write $\mathcal{G}(\Sigma, A)$ for the full subgraph of $\mathcal{G}(\Sigma)$ with vertex set A. If A is $\mathrm{Map}(\Sigma)$-invariant, then $\mathrm{Map}(\Sigma)$ also acts on $\mathcal{G}(\Sigma, A)$. We say that $\mathcal{G}(\Sigma, A)$ is *rigid* if every automorphism is induced by an element of $\mathrm{Map}(\Sigma)$.

For example, if $C_s(\Sigma)$ is the set of separating curves, then we refer to $\mathcal{G}_s(\Sigma) = \mathcal{G}(\Sigma, C_s(\Sigma))$ as the *separating curve graph.* (Note that if $g = 0$, then this is the same as $\mathcal{G}(\Sigma)$.) The results of [BrM; Ki], together with that cited for planar surfaces, tell us $\mathcal{G}_s(\Sigma)$ is rigid if $g(\Sigma) \geq 3$ or ($g(\Sigma) = 2$ and $p(\Sigma) \geq 2$) or ($g(\Sigma) = 1$ and $p(\Sigma) \geq 2$) or ($g(\Sigma) = 0$ and $p(\Sigma) \geq 5$).

Received June 29, 2015. Revision received April 26, 2016.