Michigan Math. J. 65 (2016), 761–797

Topological Obstructions for Rational Cuspidal Curves in Hirzebruch Surfaces

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Abstract. We study rational cuspidal curves in Hirzebruch surfaces. We provide two obstructions for the existence of rational cuspidal curves in Hirzebruch surfaces with prescribed types of singular points. The first result comes from Heegaard Floer theory and is a generalization of a result by Livingston and the first author. The second criterion is obtained by comparing the spectrum of a suitably defined link at infinity of a curve with spectra of its singular points.

1. Introduction

Let C be a reduced and irreducible algebraic curve in a smooth complex surface X. A singular point p on C is called a *cusp* if it is locally irreducible. The curve is called *cuspidal* if all its singularities are cusps.

Cuspidal curves in the projective plane have been investigated in classical algebraic geometry and have been subject of intense study the past three decades. The renewed interest in these curves in the 1980s came after results by Lin and Zaidenberg [19] and Matsuoka and Sakai [21]. Moreover, two questions about plane cuspidal curves were asked by Sakai in 1994 (see [15]), and ever since, several attempts have been made to describe and classify rational cuspidal curves in the projective plane (see [6; 10; 11; 12; 13; 18; 20; 32; 33; 34; 36; 37]).

In [23] the second author turned the attention to cuspidal curves in Hirzebruch surfaces and found that many of the results for plane cuspidal curves could be extended to curves in Hirzebruch surfaces (see [24; 25]). Indeed, this does not come as a surprise since the Hirzebruch surfaces are linked to each other and the projective plane by birational transformations and since such transformations clearly transform rational curves to rational curves. However, the picture is somewhat more complicated; in general, a cuspidal curve might acquire some multibranched singular points under a birational transformation. Therefore, there is no direct correspondence between rational cuspidal curves in $\mathbb{C}P^2$ and rational cuspidal curves in Hirzebruch surfaces.

In the present article we continue this work and extend two results from the plane case to the case of cuspidal curves in Hirzebruch surfaces. The first result, given in Theorem 1.1, is a consequence of Heegaard Floer theory, and it is a generalization of the result by Livingston and the first author [3]. We refer to Section 2 for explaining notation used in the theorem and especially to Section 2.2 for the definition of the function R.

Received June 16, 2015. Revision received October 12, 2015.