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Spectral Triples from Stationary Bratteli Diagrams

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Abstract. We define spectral triples for stationary Bratteli diagrams and study associated zeta functions, traces of heat kernels, and their spectral states. We observe that the zeta functions are periodic with purely imaginary periods and that the Seeley coefficients are $\log(t)$ periodic. We interpret these as a sign of self-similarity. We describe several examples and emphasize the case of substitution tiling spaces. For such tilings, the spectral measure turns out to be the unique measure that is ergodic under the translation action.

1. Introduction

Even though noncommutative geometry [4] was invented to describe (virtual) noncommutative spaces, it turned out also to provide new perspectives on (classical) commutative spaces. In particular, Connes' idea of spectral triples aiming at a spectral description of geometry has generated new concepts, or shed new light on existing ones, for topological spaces: dimension spectrum, Seeley type coefficients, spectral state are notions derived from the spectral triple, and we will talk about them here. Indeed, we study in this paper certain spectral triples for commutative algebras that are associated with stationary Bratteli diagrams, that is, with the space of infinite paths on a finite oriented graph. Such Bratteli diagrams occur in systems with self-similarity such as the tiling systems defined by substitutions.

Our construction follows from earlier ones for metric spaces, which go under the name "direct sum of point pairs" [3] or "approximating graph" [15], suitably adapted to incorporate the self-similar symmetry. The construction is therefore more rigid. The so-called Dirac operator D of the spectral triple will depend on a parameter ρ, which is related to the self-similar scaling. We observe a new feature that, we believe, ought to be interpreted as a sign of self-similarity: The zeta function is periodic with purely imaginary period $\frac{2\pi i}{\log \rho}$. Correspondingly, what corresponds to the Seeley coefficients (in the case of manifolds) in the expansion of the trace of the heat-kernel e^{-tD^2} is here given by functions of $\log t$ that are $\frac{2\pi}{\log \rho}$-periodic. This has consequences for the usual formulae for tensor products of spectral triples. If we take the tensor product of two such triples and compare the spectral states $\mathcal{T}_{1,2}$ for the individual factors with the spectral state $\mathcal{T}$ of the tensor product, then a formula like $\mathcal{T}(A_1 \otimes A_2) = \mathcal{T}_1(A_1)\mathcal{T}_2(A_2)$ will not always hold due to resonance phenomena of the involved periodicities.

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