Michigan Math. J. 65 (2016), 703–714

Cut Limits on Hyperbolic Extensions

PEDRO ONTANEDA

ABSTRACT. Hyperbolic extensions were defined and studied in [4]. Cut limits of families of metrics were introduced in [5]. In this paper, we show that if a family of metrics $\{h_\lambda\}$ has cut limits, then the family of hyperbolic extensions $\{\mathcal{E}_k(h_\lambda)\}$ also has cut limits.

The results in this paper are used in the problem of smoothing Charney–Davis strict hyperbolizations [2; 3].

1. Introduction

This paper deals with the relationship between two concepts: "hyperbolic extensions", which were studied in [4], and "cut limits of families of metrics", which were defined in [5]. Before stating our main result, we first introduce these concepts here.

1.1. Hyperbolic Extensions

Recall that the hyperbolic n-space $\mathbb{H}^n$ is isometric to $\mathbb{H}^k \times \mathbb{H}^{n-k}$ with warp product metric $(\cosh^2 r)\sigma_{\mathbb{H}^k} + \sigma_{\mathbb{H}^{n-k}}$, where $\sigma_{\mathbb{H}^l}$ denotes the hyperbolic metric of $\mathbb{H}^l$, and $r : \mathbb{H}^{n-k} \to [0,\infty)$ is the distance to a fixed point in $\mathbb{H}^{n-k}$. For instance, in the case $n = 2$, since $\mathbb{H}^1 = \mathbb{R}^1$, we have that $\mathbb{H}^2$ is isometric to $\mathbb{R}^2 = \{(u,v)\}$ with metric $\cosh^2 v\, du^2 + dv^2$. The concept of "hyperbolic extension" is a generalization of this construction; we explain this in the next paragraph.

Let (M^n, h) be a complete Riemannian manifold with *center* $o = o_M \in M$, that is, the exponential map $\exp_o : T_oM \to M$ is a diffeomorphism. The warp product metric

$$f = (\cosh^2 r)\sigma_{\mathbb{H}^k} + h$$

on $\mathbb{H}^k \times M$ is the *hyperbolic extension* (*of dimension* k) of the metric h. Here r is the distance-to-o function on M. We write $\mathcal{E}_k(M) = (\mathbb{H}^k \times M, f)$ and $f = \mathcal{E}_k(h)$. We also say that $\mathcal{E}_k(M)$ is the *hyperbolic extension* (*of dimension* k) *of* (M,h) (or just of M). Hence, for instance, we have $\mathcal{E}_k(\mathbb{H}^l) = \mathbb{H}^{k+l}$. Also, write $\mathbb{H}^k = \mathbb{H}^k \times \{o_M\} \subset \mathcal{E}_k(M)$, and we have that any $p \in \mathbb{H}^k$ is a center of $\mathcal{E}_k(M)$ (see Remarks 2.3 (3)).

REMARKS 1.1.

1. Let M^n have center o. Using a fixed orthonormal basis on T_oM and the exponential map, we can identify M with $\mathbb{R}^n$, and $M - \{o\}$ with $\mathbb{R}^n - \{0\} =$

Received June 23, 2014. Revision received June 23, 2016.

The author was partially supported by an NSF grant.