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Deforming an ε-Close to Hyperbolic Metric to a Warped Product

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Abstract. We show how to deform a metric of the form $g = g_r + dr^2$ to a warped product $\mathcal{W}g = \sinh^2(r)g' + dr^2$ (g' does not depend on r) for r less than some fixed r_0. Our main result establishes to what extent the *warp forced metric* $\mathcal{W}g$ is *close to being hyperbolic* if we assume g to be close to hyperbolic.

Introduction

We first introduce some notation. The canonical flat metric on $\mathbb{R}^k$ and the round metric on $\mathbb{S}^k$ will be denoted by $\sigma_{\mathbb{R}^k}$ and $\sigma_{\mathbb{S}^k}$, respectively. Let (M^n, g) be a complete Riemannian manifold *with center* $o \in M$, that is, the exponential map $\exp_o : T_oM \to M$ is a diffeomorphism. Using the exponential map $\exp_o$, we shall sometimes identify M with $\mathbb{R}^n$, and thus we can write the metric g on $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where r is the distance to o. The open ball of radius r in M, centered at o, will be denoted by $B_r = B_r(M)$, and the closed ball by $\bar{B}_r$. We fix a function $\rho : \mathbb{R} \to [0, 1]$ with $\rho(t) = 0$ for $t \le 0$, $\rho(t) = 1$ for $t \ge 1$, and ρ constant near 0 and 1.

Let M have center o and metric $g = g_r + dr^2$. Fix $r_0 > 0$. We define the metric $\bar{g}_{r_0}$ on $M - \{o\}$ by

$$\bar{g}_{r_0} = \sinh^2(r)\left(\frac{1}{\sinh^2(r_0)}\right)g_{r_0} + dr^2.$$

Note that this metric is a warped product (warped by sinh). Note also that to define $\bar{g}_{r_0}$ we are using the identification $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ given by the original metric g. We now force the metric g to be equal to $\bar{g}_{r_0}$ on $\bar{B}_{r_0} = \bar{B}_{r_0}(M)$ and stay equal to g outside $B_{r_0+1/2}$. For this, we define the *warp forced (on B_{r_0}) metric* as

$$\mathcal{W}_{r_0}g = \rho_{r_0}\bar{g}_{r_0} + (1 - \rho_{r_0})g,$$

where $\rho_{r_0}(t) = \rho(2t - 2r_0)$. Hence, we have

$$\mathcal{W}_{r_0}g = \begin{cases} \bar{g}_{r_0} & \text{on } \bar{B}_{r_0}, \\ g & \text{outside } B_{r_0+\frac{1}{2}}. \end{cases} \tag{0.1}$$

We call the process $g \mapsto \mathcal{W}_{r_0}g$ *warp forcing*. Note that if we choose g to be the warped-by-sinh hyperbolic metric $g = \sinh^2(t)\sigma_{\mathbb{S}^{n-1}} + dt^2$, then $\mathcal{W}_{r_0}g = g$. This suggests that if g is in some sense close to being hyperbolic, then $\mathcal{W}_{r_0}g$

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