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Differentiability Inside Sets with Minkowski Dimension One

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Abstract. We investigate Minkowski, or box-counting, dimension of universal differentiability sets of Lipschitz functions. Whilst existing results concern the Lebesgue measure and Hausdorff dimension of these fractal sets, the Minkowski dimension is stronger than Hausdorff, and we demonstrate that the lower bound one on Minkowski dimension is tight for any Euclidean space. Spaces other than the real line allow for a further refinement of the bound: the 1-Hausdorff measure of such sets must be infinite.

1. Introduction

Background and Overview of Main Results

In the present paper, we answer a natural question pointed out by Olsen in 2009, whether there is a universal differentiability set of Minkowski dimension one. Our answer is affirmative: a compact universal differentiability set with upper and lower Minkowski dimension one in $\mathbb{R}^d$, for all d, is constructed explicitly. Namely, we prove a stronger statement:

Theorem (Theorem 5.6(1)). *For every $d \geq 1$, there exists a compact subset $S \subseteq \mathbb{R}^d$ of Minkowski dimension one such that for any Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$, the set of points $x \in S$ such that g is Fréchet differentiable at x is a dense subset of S.*

Recall that Lipschitz functions on Banach spaces have rather strong differentiability properties. The classical Rademacher theorem says that Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}$ are differentiable almost everywhere with respect to the Lebesgue measure. For $d = 1$, the converse statement also holds: Each subset N of $\mathbb{R}$ with Lebesgue measure zero admits a Lipschitz function nowhere differentiable on N; see [14; 7]. However, Preiss [11] proved that all Euclidean spaces of dimension higher than one contain Lebesgue null sets that capture a point of differentiability of every Lipschitz function on the space.

Sets containing a point of differentiability of every Lipschitz function are said to have the universal differentiability property and are called universal differentiability sets (UDS). The result of [11] has sparked a modern investigation into the nature of such sets. Clearly, the set S in Theorem 5.6 quoted is a UDS.

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