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Characterizing Linear Groups in Terms of Growth Properties

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Abstract. Residual finiteness growth measures how well approximated a group is by its finite quotients. We prove that some related growth functions characterize linearity for a class of groups including all hyperbolic groups.

1. Introduction

Given a finitely generated, residually finite group Γ and a nontrivial $\gamma \in \Gamma$, we consider three functions that measure the difficulty of verifying that γ is nontrivial through homomorphisms to finite groups. The first function $\mathrm{D}_\Gamma(\gamma)$ is the minimum $|Q|$, over all finite groups Q, such that there exists a homomorphism $\phi\colon \Gamma \to Q$ with $\phi(\gamma) \neq 1$. This function was introduced in [Bou10]. In this paper, we consider two variations of D_Γ obtained by restricting the class of groups Q. We define $\mathrm{L}_\Gamma(\gamma)$ to be the minimum $|\mathrm{GL}(n, \mathbf{F}_q)|$, over all $n \in \mathbf{N}$ and finite fields $\mathbf{F}_q$, such that there exists a homomorphism $\phi\colon \Gamma \to \mathrm{GL}(n, \mathbf{F}_q)$ with $\phi(\gamma) \neq 1$. We define $\mathrm{S}_\Gamma(\gamma)$ to be the minimal $|G|$, over all finite simple groups G, such that there exists a homomorphism $\phi\colon \Gamma \to G$ with $\phi(\gamma) \neq 1$. Note that we do not insist that any of the mentioned homomorphisms be surjective, though for D_Γ, the minimum $|Q|$ for an element γ always comes from a surjective homomorphism. Fixing a finite generating subset X for Γ, for each $m \in \mathbf{N}$, we define

$$\mathrm{F}_\Gamma(m) = \max_{\substack{\gamma \in \Gamma - \{1\},\\ \|\gamma\|_X \leq m}} \mathrm{D}_\Gamma(\gamma),$$

$$\mathrm{F}_{\Gamma,\mathrm{L}}(m) = \max_{\substack{\gamma \in \Gamma - \{1\},\\ \|\gamma\|_X \leq m}} \mathrm{L}_\Gamma(\gamma),$$

$$\mathrm{F}_{\Gamma,\mathrm{S}}(m) = \max_{\substack{\gamma \in \Gamma - \{1\},\\ \|\gamma\|_X \leq m}} \mathrm{S}_\Gamma(\gamma).$$

For two functions $f, g\colon \mathbf{N} \to \mathbf{N}$, we write $f \preceq g$ if there exists a natural number C such that $f(m) \leq Cg(Cm)$ for all m and write $f \sim g$ when $f \preceq g$ and $g \preceq f$. The dependence of the above functions on the generating subset X is mild. Specifically, if Y is another generating subset, the associated functions for X, Y satisfy the equivalence relation $\sim$ (see [Bou10, Lem 1.1]). In [BM15], we proved

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