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A Remark on the Ueno–Campana's Threefold

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Dedicated to Fabrizio Catanese on his 65th birthday

Abstract. We show that the Ueno–Campana's threefold cannot be obtained as the blow-up of any smooth threefold along a smooth center, answering negatively a question raised by Oguiso and Truong.

1. Introduction

Let $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be the complex elliptic curve of period τ. There exist exactly two elliptic curves with automorphism group bigger than $\{\pm 1\}$: these are defined respectively by the periods $\sqrt{-1}$ and the cubic root of unity $\omega := (-1 + \sqrt{-3})/2$.

We consider the diagonal action of the cyclic group generated by $\sqrt{-1}$ (resp. $-\omega$) on the product

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} \quad (\text{resp. } E_\omega \times E_\omega \times E_\omega),$$

and we denote by X_4 (resp. X_6) the minimal resolution of their quotients

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}/\langle\sqrt{-1}\rangle \quad (\text{resp. } E_\omega \times E_\omega \times E_\omega/\langle -\omega\rangle).$$

The minimal resolutions are obtained by a single blow-up at the maximal ideal of each singular point of the quotients.

The threefolds X_4 and X_6 have been extensively studied in the past. In particular, they admit an automorphism of positive entropy (e.g., see [Ogu15] for more details). The variety X_4 is now referred as the *Ueno–Campana's threefold.* It has been recently shown that X_4 and X_6 are rational. Indeed, Oguiso, and Truong [OT15] showed the rationality of X_6, and Colliot-Théléne [Col15] showed the rationality of X_4, after the work of Catanese, Oguiso, and Truong [COT14]. The unirationality of X_4 was conjectured by Ueno [Uen75], whilst Campana asked about the rationality of X_4 in [Cam11].

The aim of this note is to give a negative answer to the following question raised by Oguiso and Truong (see [Ogu15, Question 5.11] and [Tru15, Question 2]).

Question 1.1. Can X_4 or X_6 be obtained as the blow-up of $\mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^1$, or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along smooth centers?

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