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Diophantine Equations in Moderately Many Variables

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1. Introduction

Let $f_1, \dots, f_r \in \mathbb{Z}[x_1, \dots, x_n]$ be polynomials of degrees $d_1, \dots, d_r$, respectively, and let $\mathbf{f}$ denote the r-tuple of polynomials $(f_1, \dots, f_r)$. We are interested in upper bounds for the counting function

$$N(\mathbf{f}, B) := \#\{\mathbf{x} \in \mathbb{Z}^n;\ f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0, |\mathbf{x}| \le B\}.$$

(Here, and throughout the paper, $|\cdot|$ denotes the maximum norm $|\mathbf{x}| = \max\{|x_1|, \dots, |x_n|\}$.) If we assume that the polynomials f_i define a complete intersection in $\mathbb{A}^n$ of dimension $n - r \ge 0$, then we have the well-known upper bound $N(\mathbf{f}, B) \ll B^{n-r}$, which we shall refer to as the trivial bound (cf. Lemma 2.5). Heuristic arguments suggest the bound $N(\mathbf{f}, B) \ll B^{n-\mathcal{D}}$, where

$$\mathcal{D} := \sum_{i=1}^{r} d_i,$$

at least as soon as $n > \mathcal{D}$. In the special case where the polynomials f_i are homogeneous of the same degree d, a famous result by Birch establishes the heuristic upper bound and indeed an asymptotic formula, as soon as

$$n > s^* + 2^{d-1}(d-1)r(r+1). \tag{1}$$

Here, $s^* = s^*_{\mathbf{f}}$ is the dimension of the so-called Birch singular locus, the affine variety

$$\{\mathbf{x} \in \mathbb{A}^n \mid \operatorname{rank} J(\mathbf{x}) < r\},$$

where $J(\mathbf{x})$ is the Jacobian matrix of size $r \times n$ with rows formed by the gradient vectors $\nabla f_i(\mathbf{x})$. (See also recent work by Dietmann [6] and, independently, Schindler [16], where s^* is replaced by an alternative quantity, sometimes leading to a stronger result.) Birch's results have recently been extended to forms of differing degree by Browning and Heath-Brown [4].

Seeing as Birch's theorem, like most results proven with the Hardy–Littlewood circle method, requires the number of variables to be rather large, we may ask if more modest upper bounds are still available for smaller values of n. At the far end of the spectrum, the *dimension growth conjecture* of Heath-Brown and Serre leads us to expect the bound $N(\mathbf{f}, B) \ll B^{n-\rho-1+\varepsilon}$ for an r-tuple of homogeneous polynomials $\mathbf{f}$ defining an irreducible nonlinear variety of codimension $\rho \le n-2$ in $\mathbb{P}^{n-1}$. The determinant method has proved a useful tool in approaching this

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