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Classifying Finite Dimensional Cubulations of Tubular Groups

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ABSTRACT. A tubular group is a group that acts on a tree with $\mathbb{Z}^2$ vertex stabilizers and $\mathbb{Z}$ edge stabilizers. This paper develops further a criterion of Wise and determines when a tubular group acts freely on a finite dimensional CAT(0) cube complex. As a consequence, we offer a unified explanation of the failure of separability by revisiting the nonseparable 3-manifold group of Burns, Karrass, and Solitar and relating it to the work of Rubinstein and Wang. We also prove that if an immersed wall yields an infinite dimensional cubulation, then the corresponding subgroup is quadratically distorted.

1. Introduction

A *tubular group* G is a group that splits as a graph of groups with $\mathbb{Z}^2$ vertex groups and $\mathbb{Z}$ edge groups. A tubular group is the fundamental group of a graph of spaces X with each vertex space homeomorphic to a torus and each edge space homeomorphic to a cylinder. The graph of spaces X is a *tubular space.* In this paper, all tubular groups will be finitely generated, and thus tubular spaces will be compact. Examples of tubular groups were used by Brady and Bridson [1] to provide groups with isoperimetric function n^α for all α in a dense subset of $[2, \infty)$. Cashen [3] provided a method of determining when two tubular groups are quasi-isometric. Wise [9] gave a criterion that determines whether or not a tubular group acts freely on a CAT(0) cube complex and characterized which tubular groups act cocompactly on a CAT(0) cube complex. This paper determines which tubular groups act on finite dimensional CAT(0) cube complexes.

DEFINITION 1.1. An *infinite cube* in a CAT(0) cube complex is the union of an ascending sequence of n-cubes c_n of $\widetilde{X}$ such that c_n is a subcube of c_{n+1} for each n.

Wise reduces the existence of cubulations to a combinatorial criterion called *equitable sets.* Given an equitable set, we can construct a finite set of *immersed walls.* An immersed wall is a graph immersed π_1-injectively in X, such that $\widetilde{\Lambda}$ lifts to a two-sided embedding $\widetilde{\Lambda} \to \widetilde{X}$. By two-sided we mean that the image of $\widetilde{\Lambda}$ in $\widetilde{X}$ is contained in a neighborhood homeomorphic to $\widetilde{\Lambda} \times [-1, 1]$. The set of all such lifts gives a G-invariant set $\mathcal{W}$ of *walls.* The pair $(\widetilde{X}, \mathcal{W})$ is a *wallspace.* In this paper, all references to "immersed walls" will be in reference to immersed

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