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Clustering for Metric Graphs Using the p-Laplacian

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Abstract. We deal with the clustering problem in a metric graph. We look for two clusters, and to this end, we study the first nonzero eigenvalue of the p-Laplacian on a quantum graph with Newmann or Kirchoff boundary conditions on the nodes. Then, an associated eigenfunction u_p provides two sets inside the graph, $\{u_p > 0\}$ and $\{u_p < 0\}$, which define the clusters. Moreover, we describe in detail the limit cases $p \to \infty$ and $p \to 1^+$.

1. Introduction

One of the major problems for networks is that of clustering. Clustering in a network means that we want to identify dense regions of it maximizing or minimizing some criterion. Here we deal with metric graphs Γ that are graphs in which we have a length for the edges and try to identify two clusters. Our approach to find two clusters in Γ is based on the following idea: given a sign-changing function u defined on the graph, just take $A = \{u > 0\}$ and $B = \{u < 0\}$ as clusters (note that the set $\{u = 0\}$ may be nontrivial, and therefore it may happen that $A \cup B \neq \Gamma$). In this work, we take u to be an eigenfunction for some differential operator; we take the p-Laplacian $-(|u'|^{p-2}u')'$ defined on the graph and study properties of this approach. We find two extreme cases: for $p = \infty$ (this is understood as the limit as $p \to \infty$), A and B are sets that have the diameter as large as possible (each one of them has the diameter equal to $\frac{\text{diam}(\Gamma)}{2}$), whereas for $p = 1$ (understood as the limit as $p \to 1$), we find that A and B are sets with large total length and small number of "cuts" in the graph (small perimeter).

A quantum graph is a graph in which we associate a differential law with each edge and which models the interaction between the two nodes defining each edge. The use of quantum graphs (in contrast to more elementary graph models, such as simple unweighted or weighted graphs) opens up the possibility of modeling the interactions between agents identified by the graph vertices in a more detailed manner than with standard graphs. Quantum graphs are used to model thin tubular structures, so-called graph-like spaces, and they are their natural limits as the radius of a graph-like space tends to zero. On both, the graph-like spaces and the metric graphs, we can naturally define Laplace-like differential operators; see [2; 16; 26].

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