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Multiple Realizations of Varieties as Ball Quotient Compactifications

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Abstract. We study the number of distinct ways in which a smooth projective surface X can be realized as a smooth toroidal compactification of a ball quotient. It follows from work of Hirzebruch that there are infinitely many distinct ball quotients with birational smooth toroidal compactifications. We take this to its natural extreme by constructing arbitrarily large families of distinct ball quotients with biholomorphic smooth toroidal compactifications.

1. Introduction

Let $\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$ with its Bergman metric, and $\Gamma \subset \mathrm{PU}(2, 1)$ a nonuniform lattice. Then $Y = \mathbb{B}^2/\Gamma$ is a complex orbifold with a finite number of cusps, and it admits a number of compactifications by a normal projective variety, which may or may not be smooth. When Γ is neat (see Section 2.1), then Y has a particularly nice *toroidal compactification,* which is a smooth projective surface [1; 7]. It is an important and open question to decide which projective surfaces are compactifications of ball quotients and, more specifically, which smooth projective surfaces are smooth toroidal compactifications.

In this paper, we study the number of ways in which a fixed projective surface can arise as a compactification of a ball quotient. A byproduct of Hirzebruch's construction of smooth projective surfaces with c_1^2/c_2 arbitrarily close to 3 [3] is an infinite family of ball quotients with *birational* smooth toroidal compactifications (cf. [4], and see [5] for more birational examples). The purpose of this paper is to exhibit arbitrarily large families of distinct ball quotients with *biholomorphic* smooth toroidal compactifications.

Theorem 1.1. *For any natural number n, there exists a smooth projective surface $X = X(n)$ and pairwise nonisomorphic quotients $Y_1, \dots, Y_n$ of $\mathbb{B}^2$ by neat lattices $\Gamma_1, \dots, \Gamma_n$ in* $\mathrm{PU}(2, 1)$ *such that X is a smooth toroidal compactification of Y_i for each $1 \le i \le n$.*

We note that finiteness is necessary, that is, a fixed smooth projective surface can only arise as a smooth toroidal compactification in finitely many ways. Indeed, if

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