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# Arc Complexes, Sphere Complexes, and Goeritz Groups

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Abstract. We show that if a Heegaard splitting is obtained by gluing a splitting of Hempel distance at least 4 and the genus-1 splitting of $S^2 \times S^1$, then the Goeritz group of the splitting is finitely generated. To show this, we first provide a sufficient condition for a full subcomplex of the arc complex for a compact orientable surface to be contractible, which generalizes the result by Hatcher that the arc complexes are contractible. We then construct infinitely many Heegaard splittings, including the above-mentioned Heegaard splitting, for which suitably defined complexes of Haken spheres are contractible.

## Introduction

Let $\Sigma_{g,n}$ be a compact connected orientable surface of genus $g$ with $n$ holes, where $n \geq 3$ if $g = 0$ and $n \geq 1$ if $g \geq 1$. As an analogue of the curve complex, the *arc complex* $\mathcal{A}_{g,n}$ of $\Sigma_{g,n}$ is defined to be the simplicial complex whose vertices are isotopy classes of essential arcs in $\Sigma_{g,n}$ and whose $k$-simplices are collections of $k + 1$ vertices represented by pairwise disjoint and nonisotopic arcs in $\Sigma_{g,n}$. Hatcher [13] proved that the complex $\mathcal{A}_{g,n}$ is contractible. See also Cho, McCullough, and Seo [8], Irmak and McCarthy [15] and Korkmaz and Papadopoulos [18] for related works on arc complexes.

In Section 1, we provide a useful sufficient condition for a full subcomplex of the arc complex to be contractible (Theorem 1.3). Since the arc complex $\mathcal{A}_{g,n}$ itself satisfies this condition, it is contractible, which gives an updated proof for Hatcher's result. Moreover, we also show that the full subcomplex $\mathcal{A}^*_{g,n}$ of $\mathcal{A}_{g,n}$, with $n \geq 2$, spanned by vertices of arcs connecting different boundary components is contractible.

A genus-$g$ *Heegaard splitting* of a closed orientable 3-manifold $M$ is a decomposition of the manifold into two handlebodies of the same genus $g$. That is, $M = V \cup W$ and $V \cap W = \partial V = \partial W = \Sigma$, where $V$ and $W$ are handlebodies of genus $g$, and $\Sigma$ is their common boundary surface. We simply denote by $(V, W; \Sigma)$ the splitting and call the surface $\Sigma$ the *Heegaard surface* of the splitting. It is well known that every closed orientable 3-manifold admits a genus-$g$ Heegaard splitting for some genus $g \geq 0$. Given a genus-$g$ Heegaard

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