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Curves Disjoint from a Nef Divisor

JOHN LESIEUTRE & JOHN CHRISTIAN OTTEM

ABSTRACT. On a projective surface it is well known that the set of curves orthogonal to a nef line bundle is either finite or uncountable. We show that this dichotomy fails in higher dimension by constructing an effective, nef line bundle on a threefold that is trivial on countably infinitely many curves. This answers a question of Totaro. As a pleasant corollary, we exhibit a quasi-projective variety with only a countably infinite set of complete, positive-dimensional subvarieties.

1. Introduction

If L is a nef line bundle on a smooth complex projective surface, then the set of curves C such that $L \cdot C = 0$ is either finite or uncountable (when some such C moves in a positive-dimensional family). This follows essentially from the Hodge index theorem (see Section 1.1). In [7], Totaro asked whether this remains true in higher dimensions:

QUESTION. Is there a nef line bundle L on a normal complex projective variety X such that the set of curves C with $L \cdot C = 0$ is countably infinite?

In this note we construct examples of such L in any dimension greater than two, which are in fact effective and movable divisors. Perhaps the surprising thing is not that such examples exist, but that they turn out to be so accessible: our example is the blow-up of $\mathbb{P}^3$ at eight very general points, and L is the anticanonical divisor. Our main result is the following:

THEOREM 1. *There exists a smooth projective rational threefold X with nef anticanonical divisor such that the set of curves C with $-K_X \cdot C = 0$ is countably infinite and Zariski dense.*

In particular, since $-K_X$ is effective in the example, the complement of the zero set of a global section gives an example of the following:

COROLLARY 2. *There exists a quasi-projective variety with only a countably infinite set of complete, positive-dimensional subvarieties.*

We show in Corollary 7 that this is impossible in dimension less than 3. Furthermore, we show that the question has an affirmative answer even if the line bundle is required to be big and nef, which is impossible in dimension less than four (cf. Remark 13).

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