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Extremal Divisors on Moduli Spaces of Rational Curves with Marked Points

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Abstract. We study effective divisors on $\overline{M}_{0,n}$, focusing on hypertree divisors introduced by Castravet and Tevelev and on the proper transforms of divisors on $\overline{M}_{1,n-2}$ introduced by Chen and Coskun. We relate these two types of divisors and exhibit divisors on $\overline{M}_{0,n}$ for $n \geq 7$ that furnish counterexamples to a conjectural description of the effective cone of $\overline{M}_{0,n}$ given by Castravet and Tevelev.

1. Introduction

The moduli space $M_{0,n}$ parameterizes equivalence classes of n distinct marked points on $\mathbb{P}^1$ under the action of PGL_2. We will be primarily concerned with $\overline{M}_{0,n}$, the Deligne–Mumford compactification of $M_{0,n}$ by stable rational curves with n marked points. The Deligne–Mumford compactification parameterizes nodal trees of $\mathbb{P}^1$s with n markings such that each component has at least three "special" points (markings or nodes) modulo automorphisms (see Figure 1).

The locus $\overline{M}_{0,n} \setminus M_{0,n}$ is a union of boundary divisors, defined as follows: for $I \subset \{1, \dots, n\}$ with both I and $\{1, \dots, n\} \setminus I$ of size at least two, the boundary divisor δ_I consists of classes of stable rational curves in $\overline{M}_{0,n} \setminus M_{0,n}$ with a node separating the markings corresponding to indices in I and $\{1, \dots, n\} \setminus I$.

Significantly, $\overline{M}_{0,n}$ can be realized as an iterated blow-up of $\mathbb{P}^{n-3}$ via a Kapranov morphism. Any Kapranov morphism restricts to an isomorphism of $M_{0,n}$ with its image, and any boundary divisor is contracted by some Kapranov morphism. Hence, each boundary divisor generates an extremal ray of the effective cone of $\overline{M}_{0,n}$, and select boundary divisors together with the pull-back of a hyperplane class under a Kapranov morphism comprise free generators for the class group $\mathrm{Cl}(\overline{M}_{0,n})$ [K]. We will use these Kapranov generators throughout the paper.

In Section 2, we describe a method of specifying divisors on $\overline{M}_{0,n}$ via polynomials in n variables. We discuss how to compute the classes of these divisors and include Macaulay2 code to compute classes. Although useful for checking results on $\overline{M}_{0,n}$ with $n \leq 10$, the code is not practical for large n.

In Section 3, we recall the definitions of hypertrees and hypertree divisors from [CT]. A major result of [CT] is that hypertree divisors corresponding to "irreducible" hypertrees are exceptional divisors of some birational contraction and

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