Michigan Math. J. 65 (2016), 227–250

Equivariant Versal Deformations of Semistable Curves

Jarod Alper & Andrew Kresch

Abstract. We prove that given any n-pointed prestable curve C of genus g with linearly reductive automorphism group $\mathrm{Aut}(C)$, there exists an $\mathrm{Aut}(C)$-equivariant miniversal deformation of C over an affine variety W. In other words, we prove that the algebraic stack $\mathfrak{M}_{g,n}$ parameterizing n-pointed prestable curves of genus g has an étale neighborhood of $[C]$ isomorphic to the quotient stack $[W/\mathrm{Aut}(C)]$.

1. Introduction

A fundamental question in algebraic geometry is to understand the relationship between arbitrary algebraic stacks and quotient stacks. While not every algebraic stack is a quotient stack ([7] and [11]), it is natural to conjecture that every algebraic stack is étale locally a quotient stack around a point with linearly reductive stabilizer. Precisely, we formulate the conjecture as follows.

Conjecture 1.1. *Let X be an algebraic stack locally of finite type over an algebraically closed field k with separated and quasi-compact diagonal such that X has affine stabilizer groups at all closed points. Suppose $x \in X(k)$ has a linearly reductive stabilizer group scheme G_x. Then there exists an affine scheme W over k with an action of G_x, a k-point $w \in W$, and an étale, representable morphism*

$$f : [W/G_x] \to X$$

such that $f(w) = x$ and f induces an isomorphism of stabilizer groups at w.

Conjecture 1.1 after replacing W with an algebraic space is a particular case of the conjecture stated in [3]. Similar questions were raised in [5, §5] and [16, §2].

This conjecture implies that étale-local properties of general algebraic stacks (satisfying the hypotheses of Conjecture 1.1) can be inferred from properties of algebraic stacks of the form $[\mathrm{Spec}(A)/G]$ with G linearly reductive. Such quotient stacks are particularly well understood; in particular, many geometric properties of $[\mathrm{Spec}(A)/G]$ can be related to properties of the GIT quotient $\mathrm{Spec}(A^G)$. Additionally, as suggested by Rydh, it is possible to attach to an algebraic stack X satisfying Conjecture 1.1 at a point $x \in X(k)$ a Henselian localization $\mathcal{O}^h_{X,x}$ that is a comodule algebra over the Hopf algebra of G_x such that $[\mathrm{Spec}(\mathcal{O}^h_{X,x})/G_x] \to X$ satisfies analogous properties to the usual Henselization $\mathrm{Spec}(\mathcal{O}^h_{W,w}) \to W$.

Received March 25, 2014. Revision received January 27, 2016.

The first author was partially supported by the Alexander von Humboldt Foundation. The second author was partially supported by the Swiss National Science Foundation.